The University of Chicago

REDUCED POSITIVE QUATERNARY QUADRATIC FORMS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1936

By

STANMORE BROOKS TOWNES

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REDUCED POSITIVE QUATERNARY QUADRATIC FORMS

INTRODUCTION

Dickson[1] developed a geometrical method for finding certain inequalities which segregate a single form from its class and applied it to positive binary and ternary forms. Dirichlet[2] had previously introduced the idea of reduced parallelograms and parallelopipeds and sketched a more involved method for determining reduced quadratic forms. Dickson proved in full facts stated by Dirichlet.

Following the method developed by Dickson, the author has extended the work to reduced, positive, quadratic forms in four variables. It is shown in chapter i that a positive quadratic form in n variables can be represented by a point lattice in n dimensions. Use is made of certain formulae for vectors in n dimensions given by Sommerville[3]. In chapter ii, the polyhedron is described whose interior under proper conditions is the locus of points in three dimensions which are closer to the origin than to any other lattice point. In a manner analogous to that used by Dickson in three dimensions, a reduced fundamental hyper-parallelopiped in four dimensions is defined in chapter iii.

Conditions for a reduced form are given in chapter iv and it is proved that they may be imposed. In chapter v, the method used in proving that no two reduced forms are equivalent is given and reference is made to the complete proof, to a tabulation of automorphs and of reduced forms of determinant ≤ 50 which are found in the unpublished portion of the thesis.

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[1] L. E. Dickson, Studies in the Theory of Numbers, (Chicago: University of Chicago Press, 1930), chap. xi.

[2] Lejeune Dirichlet, "Über die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen", Jour. fur Math., XL (1850), pp. 209-27.

[3] D. M. Y. Sommerville, Geometry of N Dimensions, (New York: E. P. Dutton, 1929), p. 76.

CHAPTER I

REPRESENTATION OF A POSITIVE QUADRATIC FORM IN N VARIABLES BY MEANS OF A POINT LATTICE.

Let $f = \sum_{i,j=1}^{n} a_{ij} x_i x_j$, $a_{ij} = a_{ji}$ be a positive quadratic form in n variables. Let $a = |a_{ij}|$ be the determinant of f and $p_o = 1$, and p_i be the principal minor of the i^{th} order of a containing elements from the first i rows and columns, $i = 1,2,\ldots,n$. Since f is a positive form, all the p_i are positive. Let B_{ij} be the determinant formed from p_i by replacing the elements in the last column by $a_{1j}, a_{2j}, \ldots, a_{ij}$.

(1) $$B_{ii} = p_i \text{ and } B_{ij} = 0, \quad j < i$$

since in the latter case B_{ij} has two columns alike.

Employ a rectangular coordinate system in a space of n dimensions with any point O as origin. Let P have the coordinates

(2) $$(b_{1j}, b_{2j}, \ldots, b_{nj}), \quad b_{ij} = B_{ij} / \sqrt{p_{i-1} p_i}, \quad i, j = 1,2,\ldots,n.$$

Theorem 1. The length of the vector $\overline{OP_j}$ is $\sqrt{a_{jj}}$ and the cosine of the angle between any two vectors $\overline{OP_i}$ and $\overline{OP_j}$ is $a_{ij}/\sqrt{a_{ii}a_{jj}}$.

It is sufficient to show that

(3) $$\sum_{K=1}^{n} b_{Ki} b_{Kj} = a_{ij}$$

since the square of the length of $\overline{OP_j}$ is $\sum_{K=1}^{n} b_{Kj} b_{Kj}$ and if $e_{ij} < 180°$ is the angle between $\overline{OP_i}$, and OP_j, then $\cos e_{ij} = \sum_{K} b_{Ki} b_{Kj} / \overline{OP_i}\ \overline{OP_j}$.

To prove (3), let

$$A_{Kij} = \begin{vmatrix} a_{11} \ldots a_{1K}\ a_{1j} \\ \cdot \\ \cdot \\ a_{K1} \ldots a_{KK}\ a_{Kj} \\ a_{i1} \ldots a_{iK}\ 0 \end{vmatrix}$$

We shall prove by induction that

(4) $$\sum_{K=1}^{n} b_{Ki} b_{Kj} = -A_{rij} / p_r .$$

First,

$$b_{1i} b_{1j} = a_{1i} a_{1j} / p_1 = -\begin{vmatrix} a_{11} & a_{1i} \\ a_{i1} & 0 \end{vmatrix} / p_1 ,$$

hence (4) holds when $r = 1$. Assume

$$\sum_{K=1}^{s} b_{Ki} b_{Kj} = -A_{sij} / p_s ,$$

then

$$\sum_{k=1}^{s+1} b_{ki}b_{kj} = -A_{sij}/p_s - b_{s+1,j}b_{s+1,j} = -A_{sij}/p_s - B_{s+1,j}B_{s+1,j}/p_s\,p_{s+1}.$$

In the determinant $A_{s+1,ij}$, the elements in the two-rowed minor in the lower right hand corner are, respectively,

$$a_{s+1,s+1},\ a_{s+1,j},\ a_{i,s+1},\ 0.$$

The cofactors of these elements are, respectively, A_{sij}, the transpose of $-B_{s+1,i}$, $-B_{s+1,j}$ and p_{s+1}. By a theorem on determinants,

$$B_{s+1,i}\,B_{s+1,j} = p_{s+1}A_{sij} - p_s\,A_{s+1}.$$

Hence

$$-A_{sij}/p_s - B_{s+1,i}B_{s+1,j}/p_s\,p_{s+1} = -A_s/p_s - p_{s+1}A_{sij} - p_s\,A_{s+1,ij}/p_s\,p_{s+1} = -A_{s+1,ij}/p_{s+1}.$$

Therefore (4) is proved and

$$\sum_{k=1}^{n} b_{ki}b_{kj} = -A_{nij}/p_n.$$

In $-A_{nij}$ subtract the j^{th} column from the last getting in the last column all zeros except the last element which is $-a_{ij}$. Hence,

$$-A_{nij}/p_n = a_{ij}p_n/p_n = a_{ij}.$$

Therefore (3) follows.

From the properties (1) of the B_{ij}, P_1 is on the first axis on the positive side of the origin, P_2 is on the plane determined by the first two axes and has its second coordinate positive. In general, P_i is in the linear space determined by the first i axes and the i^{th} coordinate of P_i is positive and equal to $\sqrt{p_i/p_{i-1}}$. Hence the hyper-volume of the hyper-parallelopiped of i dimensions, $i \leq n$, determined by the vectors $\overline{OP}_1, \ldots, \overline{OP}_i$ is the product of

$$\sqrt{p_1/p_o} \cdot \sqrt{p_2/p_1} \cdot \ldots \cdot \sqrt{p_i/p_{i-1}} = \sqrt{p_i}.$$

Let V_i represent the vector $\overline{OP}_i$.

Theorem 2. For all sets of integers $x_1, x_2, \ldots, x_n$, the points with the vectors $V = \sum_{i=1}^{n} x_i V_i$ form a point lattice L.

The point with the vector V has the coordinates $\xi_1 \ldots \xi_n$ where

$$\xi_i = \sum_{j=1}^{n} x_j B_{ij}/\sqrt{p_{i-1}p_i}. \tag{6}$$

The square of the length of the vector V is represented by

$$\sum_{i=1}^{n} \xi_i^2 = \sum_{i,j=1}^{n} x_i x_j \sum_{k=1}^{n} b_{ki}b_{kj} = f, \tag{7}$$

by (3).

Let

$$V_i' = \sum_{j=1}^{n} \alpha_{ji} V_j,\ i = 1, \ldots, n. \tag{8}$$

Then

$$\sum_{i=1}^{n} x_i' V_i' = \sum_{j=1}^{n} x_j V_j \tag{9}$$

holds if and only if

$$x_i = \sum_{j=1}^{n} \alpha_{ij} x_j', \quad i = 1, \ldots, n. \tag{10}$$

Let P_i' be the point such that the vector $\overline{OP_i'}$ is V_i', $i = 1, \ldots, n$. The fundamental hyper-parallelopiped determined by the vectors V_i' is a hyper-parallelopiped of a point lattice L_1.

Henceforth let α_{ij} be integers. For any integer x_i', (10) yields integers x_{ij}. Hence every point of L_1 is a point of L. The converse is true if and only if the determinant $|\alpha_{ij}| = \pm 1$, since for any integers x_i, (10) then yields integers x_i'. Then L and L_1 coincide.

Theorem 3. To any transformation (10) of determinant ± 1 corresponds a change from one fundamental hyper-parallelopiped to another of any point lattice and conversely.

Let $f' = \sum_{i,j=1}^{n} a_{ij}' x_i' x_j'$ be the form f becomes on undergoing the transformation (10), where $a_{rs}' = \sum_{i,j=1}^{n} \alpha_{ir} \alpha_{js} a_{ij}$.

From (6) the k^{th} coordinate of P_s' is

$$\sum_{j=1}^{n} \alpha_{js} b_{kj} = b_{ks}'. \tag{11}$$

Now

$$\sum_{k=1}^{n} b_{kr}' b_{ks}' = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{js} b_{ks} \alpha_{ir} b_{ki} = \sum_{i,j=1}^{n} \alpha_{ir} \alpha_{js} \sum_{k=1}^{n} b_{kj} b_{ki} = \sum_{i,j=1}^{n} \alpha_{ir} \alpha_{js} a_{ij} = a_{rs}'.$$

Hence $\cos e_{rs}' = a_{rs}' / \sqrt{a_{rr}' a_{ss}'}$ and the length of $\overline{OP_s'} = a_{ss}'$. Thus we have proved

Theorem 4. Let an integral transformation (10) of determinant ± 1 replace f by f'. Then f and f' determine the same point lattice. In other words, all properly or improperly equivalent positive quadratic forms in n variables are represented by the same point lattice.

CHAPTER II

SOME THEOREMS REGARDING POINT LATTICES IN THREE DIMENSIONS

Section 1. Introduction

Determine a reduced fundamental parallelopiped[1] of any point lattice L of three dimensions as follows. Start with any lattice point O and choose three pairs of lattice points P,P'; Q,Q' and M,M' such that O bisects the segment between each pair. Choose P such that OP is a minimum. Designate by Q the lattice point nearest O and not on OP. Let M be the lattice point nearest O not lying in the plane POQ. Select Q and M so that the angles QOM, POM, POQ are all less than 90°, or all equal to or greater than 90°. That this is possible is shown by Dickson[2].

Now define a,b,c,r,s,t as follows:

$$a = \overline{OP}^2 \qquad b = \overline{OQ}^2 \qquad c = \overline{OM}^2 \qquad r = \sqrt{bc}\cos QOM$$

$$s = \sqrt{ac}\cos POM \qquad t = \sqrt{ab}\cos POQ. \tag{12}$$

For all sets of integers x, y, z, the points W with the vectors $\overline{OW} = x\overline{OP} + y\overline{OQ} + z\overline{OM}$ form the point lattice L.

The coordinates of W referred to the rectangular system of chapter i are

$$\begin{matrix} xb_{11} + yb_{12} + zb_{13} \\ xb_{21} + yb_{22} + zb_{23} \\ xb_{31} + yb_{32} + zb_{33} \end{matrix} \tag{13}$$

where the b_{ij} have the definition given in (2) of chapter i with the a_{ij} replaced by the element in the i^{th} row j^{th} column of the matrix

$$\begin{Vmatrix} a & t & s \\ t & b & r \\ s & r & c \end{Vmatrix}.$$

The square of the distance $\overline{OW}$ is given by

$$ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy. \tag{14}$$

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[1] L. E. Dickson, Studies in the Theory of Numbers, (Chicago: University of Chicago Press, 1930), p. 162.

[2] Ibid., p. 164.

We shall refer to (x,y,z) as the oblique coordinates of a lattice point W. The small letter w(x,y,z) will designate the plane which is the perpendicular bisector of the line OW. The coordinates of every point will be given in the oblique system except when expressly stated otherwise. The equation of every plane will be in the rectangular system.

From the manner of selection of P, Q, M

(15) $$c \geq b \geq a$$

and either

(16) $$r > 0,\ s > 0,\ t > 0$$

or

(17) $$r \leq 0,\ s \leq 0,\ t \leq 0.$$

Since the parallelopiped P,Q,M is reduced

(18) $$a - 2|s| \geq 0,\ a - 2|t| \geq 0,\ b - 2|r| \geq 0$$

and

(19) $$a + b + 2r + 2s + 2t \geq 0.$$

By Dickson we may select our reduced parallelopiped so that if $a = b = c$, then $a+2r$ and $a+2s$ are not both zero.

Dickson showed that

(20) $$abc/2 + 2rst - ar^2 - bs^2 - ct^2 \geq 0$$

and that

(21) $$ab^2/2 + 2rst - ar^2 - bs^2 - bt^2 \geq 0.$$

The lattice points in the plane POQ belonging to fundamental parallelograms having O as one vertex will be designated

P(1,0,0), R(1,1,0), Q(0,1,0), S(-1,1,0),
P'(-1,0,0), R'(-1,-1,0), Q'(0,-1,0), S'(1,-1,0).

The subscript 1 will designate the corresponding lattice point in the plane containing M and parallel to POQ.

If r, s and t are each positive, then the traces of the planes p, q, s, p', q', s' on the plane POQ determine the hexagon H whose interior is the locus of all points in the plane POQ closer to O than to any other lattice point, according to Dickson. If r, s and t are negative or zero, then planes s ans s' are replaced by r and r', respectively.

Section 2. The Polyhedrons Π_i.

We shall now describe certain polyhedrons. The interior of each of these under proper conditions will be shown to be the locus of points in three dimensions which are closer to O than to any other lattice point. The conditions may first be subdivided

into two cases, case I, $r > 0$, $s > 0$, $t > 0$ and case II, $r \leq 0, s \leq 0$, $t \leq 0$. Under case I occur three polyhedrons, Π_1, Π_2 and Π_3, which under proper additional conditions will be shown to be the required locus. Similarly, under case II there is one such polyhedron, Π_4.

Case I. $r > 0$, $s > 0$, $t > 0$.

Each of the prism-like figures Π_i, $i = 1, 2, 3$, has as lateral faces the six planes p, q, s, p', q' and s' whose intersections with the plane POQ determine the hexagon H.

The description of the polyhedrons Π_i is completed by listing in order the vertices of each face above the plane POQ. Since the vertices are symmetrically located with respect to O, those below the plane POQ are not listed. The vertex mqp is the center of the sphere circumscribed about the tetrahedron OMPQ. Likewise, the other vertices are the centers of spheres circumscribing the corresponding tetrahedrons. The ρ_i listed after each vertex represents its distance from O, that is, the radius of the sphere circumscribing the corresponding tetrahedron. The tetrahedrons corresponding to vertices with the same ρ_i listed after them may be shown by elementary geometry to be either congruent or symmetric, hence to have the same circumradius. These will presently be calculated.

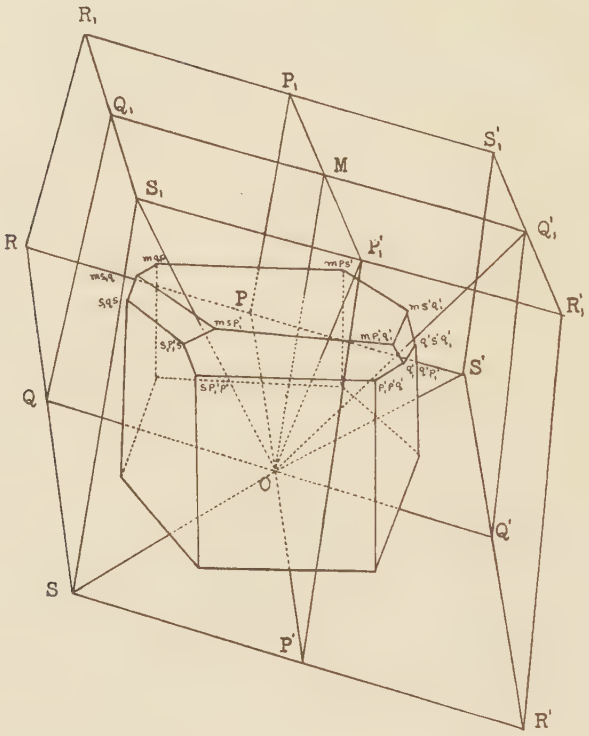
In the following tabulation, vertices, in order, of each face above the plane POQ are listed. Reference to the drawing of Π_1 will make the notation clearer.

Case I. $r > 0$, $s > 0$, $t > 0$.

A. If $r \leq s$, $r \leq t$, it will be proved that the required locus is

Π_1

Face m	Face p_1'	Face q_1'	Face s_1
m q p ρ_0	p_1'm q_1' ρ_0	q_1'm p_1' ρ_0	s_1 s p' ρ_4
m p s' ρ_1	$p_1'q_1'q'$ ρ_4	$q_1'p_1'q'$ ρ_4	s_1 p'm ρ_1
m s'q_1' ρ_4	$p_1'q'p'$ ρ_1	$q_1'q's'$ ρ_0	s_1 m q ρ_4
m $q_1'p_1'$ ρ_0	$p_1'p's$ ρ_0	$q_1's'$m ρ_4	s_1 q s ρ_1
m $p_1's_1'$ ρ_1	$p_1's$ s_1 ρ_4		
m s_1'q ρ_4	$p_1's_1$ m ρ_1		



UPPER HALF OF Π_1

B. If $s \leq r$, $s \leq t$, it will be proved that the required locus is

Π_2

Face m	Face p_1'	Face q_1'	Face s_1'
m q p ρ_0	p_1'm q_1' ρ_0	q_1'm p_1' ρ_0	s_1's'q_1' ρ_5
m p s_1' ρ_5	$p_1'q_1'p'$ ρ_5	$q_1'p_1'p'$ ρ_5	$s_1'q_1'$m ρ_2
m $s_1'q_1'$ ρ_2	$p_1'p's$ ρ_0	$q_1'p'q'$ ρ_2	s_1'm p ρ_5
m $q_1'p_1'$ ρ_0	$p_1's$ m ρ_5	$q_1'q's'$ ρ_0	s_1'p s' ρ_2
m $p_1's$ ρ_5		$q_1's's_1'$ ρ_5	
m s q ρ_2		$q_1's_1'$m ρ_2	

C. If $t \leq r$, $t \leq s$, it will be proved that the required locus is

Π_3

Face m	Face p_1'	Face q_1'	Face r_1'
m q p ρ_0	p_1'm q_1' ρ_0	q_1'm p_1' ρ_0	$r_1'p'p_1'$ ρ_3
m p q_1' ρ_3	$p_1'q_1'r_1'$ ρ_4	$q_1'p_1'r_1'$ ρ_4	$r_1'p_1'q'$ ρ_4
m $q_1'p_1'$ ρ_0	$p_1'r_1'p'$ ρ_3	$q_1'r_1'q'$ ρ_3	$r_1'q'q_1'$ ρ_3
m p_1'q ρ_3	$p_1'p's$ ρ_0	$q_1'q's'$ ρ_0	$r_1'q_1'p'$ ρ_4
	$p_1's$ q ρ_4	$q_1's'p$ ρ_4	
	p_1'q m ρ_3	$q_1'p$ m ρ_3	

These figures are convex. Each consists of 14 faces, 36 edges and 24 vertices. In each the number of edges exceeds by two the sum of the number of faces and vertices, hence, by Euler's formula, each is a closed polyhedron.

The tetrahedrons OPMQ, OPSM,...., the center of whose circumscribing spheres are the vertices of the polyhedrons may be shown by elementary geometry to have the same volume. The volume of OPQM is one third the volume of the fundamental parallelopiped determined by OP, OQ and OM. Hence, by (5) of chapter 1, the volume of OPQM is $\sqrt{\Delta}/3$, where Δ is the determinant of the form (14).

If the product of the pairs of opposite edges of a tetrahedron are respectively e_1, e_2, e_3, and its volume V, then its circumradius ρ is given by

(22) $$\sqrt{4e_1^2e_2^2 - (e_1^2 + e_2^2 - e_3^2)^2}/24\ V.$$

In OPQM,

$\overline{OP}^2 = a$, $\overline{OQ}^2 = b$, $\overline{OM}^2 = c$, $\overline{QM}^2 = b-2r+c$, $\overline{PM}^2 = a-2s+c$, $\overline{PQ}^2 = a-2t+b$ by (14). Substituting these values in (22) gives the value of $4\Delta\rho_0^2$. The value of $4\Delta\rho_i^2$, $i = 1,\ldots,6$, is obtained in a similar manner.

(23)
$$\begin{aligned}
4\Delta\rho_0^2 &= abc(a+b+c-2r-2s-2t)-a^2r^2-b^2s^2-c^2t^2+2abrs+2bcst+2acrt.\\
4\Delta\rho_1^2 &= abc(a+b+c+2r-2s-2t)-(bs+ct+ar-2st)^2.\\
4\Delta\rho_2^2 &= abc(a+b+c-2r+2s-2t)-(ar+ct+bs-2rt)^2.\\
4\Delta\rho_3^2 &= abc(a+b+c-2r-2s+2t)-(ar+bs+ct-2rs)^2.\\
4\Delta\rho_4^2 &= abc(a+b+c+2r-2s-2t)-a^2r^2-b^2s^2-c^2t^2-2abrs-2acrt+2bcst\\
&\qquad -4r(s+t-r)(bc-ar-bs-ct).\\
4\Delta\rho_5^2 &= abc(a+b+c-2r+2s-2t)-a^2r^2-b^2s^2-c^2t^2-2abrs-2bcst+2acrt\\
&\qquad -4s(r+t-s)(ac-ar-bs-ct).\\
4\Delta\rho_6^2 &= abc(a+b+c-2r-2s+2t)-a^2r^2-b^2s^2-c^2t^2+2abrs-2bcst-2acrt\\
&\qquad -4t(r+s-t)(ab-ar-bs-ct).
\end{aligned}$$

Theorem 5. If $r < s$, $r < t$, then $\rho_4 > \rho_1$.
Theorem 6. If $s < r$, $s < t$, then $\rho_5 > \rho_2$.
Theorem 7. ρ_4, ρ_5 and ρ_6 are each greater than ρ_0.
Theorem 8. If $s = t$, then $\rho_6 = \rho_3 = \rho_2 = \rho_5$.
Theorem 9. If $r = t$, then $\rho_1 = \rho_3 = \rho_4 = \rho_6$.
Theorem 10. If $r = s$, then $\rho_1 = \rho_2 = \rho_4 = \rho_5$.
Theorem 11. If $r = s = t$, then $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = \rho_6$.

Since

$$\Delta(\rho_4^2 - \rho_1^2) = (s-r)(t-r)[(a-t)(c-2s)+(s-r)(a-t+r)+(c-s)(b-a)]$$

is positive if $r < s$, $r < t$ and zero if $r = s$ or if $r = t$, then theorem 5 and parts of theorems 8 and 9 follow. Similarly from

$$\Delta(\rho_5^2 - \rho_2^2) = (t-s)(r-s)[(a-2t)(b-r)+(t-s)(b-r+s)+(c-b)(a-t)]$$

follow theorem 6 and parts of theorem 8 and 10.

The expression $\Delta(\rho_4^2 - \rho_0^2)$ may be written

$$r(a+r-s-t)[(bc-bs-ct)+r(s+t-r)].$$

The parenthesis $(bc-bs-ct)$ is positive or zero and $r(s+t-r)$ is positive if $r < s+t$. And the expression $bs+ct \leq bc/2$ and $r(r-s-t) < bc/2$ if $r < s+t$. Hence, in either case, $\Delta(\rho_4^2 - \rho_0^2) > 0$. By similar reasoning the following two expressions are always positive:

$$\Delta(\rho_5^2 - \rho_0^2) = s(b+s-r-t)[(ac-ar-ct)+s(r+t-s)],$$
$$\Delta(\rho_6^2 - \rho_0^2) = t(c+t-r-s)[(ab-ar-bs)+t(r+s-t)].$$

Hence, theorem 7 follows.

The following expressions contain the factors listed:

	Factor
$\Delta(\rho_5^2 - \rho_6^2)$	$(s-t)$
$\Delta(\rho_4^2 - \rho_6^2)$	$(r-t)$
$\Delta(\rho_4^2 - \rho_5^2)$	$(r-s)$
$\Delta(\rho_6^2 - \rho_3^2)$	$(r-t)(s-t)$.

Theorems 8, 9, 10 and 11 follow from the foregoing.

As a consequence of theorems 5-10, if $s = t$, it follows that Π_2 and Π_3 coincide and reduce to the following polyhedron:

Face m	Face p_1'	Face q_1'
m q p ρ_0	p_1'm q_1' ρ_0	q_1'm p_1' ρ_0
m p q_1' ρ_3	$p_1'q_1'p'$ ρ_5	$q_1'p_1'p'$ ρ_5
m $q_1'p_1'$ ρ_0	$p_1'p's$ ρ_0	$q_1'p'q'$ ρ_2
m $p_1'q$ ρ_3	$p_1's$ m ρ_5	$q_1'q's'$ ρ_0
		$q_1's'p$ ρ_5
		$q_1'p$ m ρ_2

Similarly, if $r = t$, π_1 and π_3 coincide and reduce to

Face m		Face p_1'		Face q_1'	
m q p	ρ_0	p_1'm q_1'	ρ_0	q_1'm p_1'	ρ_0
m p q_1'	ρ_3	$p_1'q_1'q'$	ρ_4	$q_1'p_1'q'$	ρ_4
m $q_1'p_1'$	ρ_0	$p_1'q'p'$	ρ_3	$q_1'q's'$	ρ_0
m p_1'q	ρ_3	$p_1'p's$	ρ_0	$q_1's'$m	ρ_4
		$p_1's$ q	ρ_4		
		p_1'q m	ρ_3		

And if $r = s$, π_1 and π_2 coincide and reduce to

Face m		Face p_1'		Face q_1'	
m q p	ρ_0	p_1'm q_1'	ρ_0	q_1'm p_1'	ρ_0
m p s'	ρ_1	$p_1'q_1'p'$	ρ_5	$q_1'p_1'q'$	ρ_4
m s'q_1'	ρ_4	$p_1'p's$	ρ_0	$q_1'q's'$	ρ_0
m $q_1'p_1'$	ρ_0	$p_1's$ m	ρ_5	$q_1's'$m	ρ_4
m $p_1's$	ρ_1				
m s q	ρ_4				

Each of these polyhedrons has 12 faces, 18 vertices and 28 edges, hence it is closed.

Finally, if $r = s = t$, π_1, π_2 and π_3 coincide and reduce to

Face m		Face p_1'		Face q_1'	
m q p	ρ_0	p_1'm q_1'	ρ_0	q_1'm p_1'	ρ_0
m p q_1'	ρ_3	$p_1'q_1'p'$	ρ_5	$q_1'p_1'q'$	ρ_4
m $q_1'p_1'$	ρ_0	$p_1'p's$	ρ_0	$q_1'q's'$	ρ_0
m p_1'q	ρ_3	$p_1's$ m	ρ_5	$q_1's'$m	ρ_4

This consists of 12 faces, 14 vertices and 24 edges, hence it is closed.

Let

π be π_1 if $r \leq s$, $r \leq t$;

π be π_2 if $s \leq t$, $s \leq r$;

π be π_3 if $t \leq r$, $t \leq s$.

Theorem 12. The locus of points closer to 0 than to any other lattice point is the interior of the polyhedron π.

Since the faces of π are the perpendicular bisectors of lines joining 0 to other lattice points, every point outside of π is closer to some other lattice point than it is to 0. In order to prove that every point inside of π is closer to 0 than to any other lattice point, it is sufficient to prove that no plane which is the perpendicular bisector of the line joining

0 to any other lattice point crosses the interior of Π. In order to prove this we will prove a series of lemmas.

Lemma 1. If W is any lattice point in one of the planes POQ, POM, QOM, then the plane w does not touch Π unless it contains one of the faces of Π.

Suppose W in the plane POQ. The parallelogram determined by OP and OQ is reduced, hence the locus of all points in the plane POQ closer to 0 than to any other lattice point in that plane is the interior of the hexagon H whose sides are the traces of planes p, q, s, p', s', q' on the plane POQ. Since the planes p, q, s, p', q', s' and w are perpendicular to the plane POQ, the plane w does not intersect the polyhedron Π. Since the parallelograms determined by OP and OM, OQ and OM are likewise reduced, analogous proofs hold for points in the planes POM and QOM.

Lemma 2. The plane v(1,1,1) does not intersect Π.

In order to prove this lemma it will be shown that OV is greater than twice the distance from 0 to the furthest vertex of Π. That is, that

$$OV > 2\rho_4 \text{ if } r \leq s,\ r \leq t;$$
$$OV > 2\rho_5 \text{ if } s \leq t,\ s \leq r;$$
$$OV > 2\rho_6 \text{ if } t \leq s,\ t \leq r;$$
$$OV > 2\rho_3 \text{ if } t \leq s,\ t \leq r.$$

By (14),

$$\overline{OV}^2 = a+b+c+2r+2s+2t.$$

If $r \leq s$, $r \leq t$,

$$\Delta(\overline{OV}^2-4\rho_4^2) = 2(s+t)(abc+4rst-2ar^2-2bs^2-2ct^2)+bc(s+t-2r)(a-s-t+2r)+ac(t-r)(b-t+r)+ab(s-r)(c-s+r)+2(s-r)(t-r)[t(c-2r)+s(b-2r)+r(a+2r)]+2bs(s-r)^2+2ct(t-r)^2+4abcr.$$

The first term is positive or zero by (20), none of the remaining terms are negative and the last term is positive, hence,

$$(\overline{OV}^2-4\rho_4^2) > 0.$$

If $s \leq r$, $s \leq t$,

$$\Delta(\overline{OV}^2-4\rho_5^2) = 2(r+t)(abc+4rst-2ar^2-2bs^2-2ct^2)+ac(r+t-2s)(b-r-t+2s)+bc(t-s)(a-s+t)+ab(r-s)(c-r+s)+2(t-s)(r-s)[t(c-2s)+r(a-2s)-s(b+2s)]+2ar(r-s)^2+2ct(t-s)^2+4abcs.$$

The first term is positive or zero by (20), none of the remaining terms are negative and the last term is positive, hence

$$(\overline{OV}^2-4\rho_5^2) > 0.$$

If $t \leq r$, $t \leq s$,

$$\Delta(\overline{OV}^2 - 4\rho_6^2) = 2(r+s)(abc+4rst-2ar^2-2bs^2-2ct^2)+ab(r+s-2t)(c-r-s+2t)+ac(r-t)(b-r+t)+bc(s-t)(a-s+t)+2(s-t)(r-t)[s(b-2t)+r(a-2t)+t(c+2t)]+2ar(r-t)^2+2bs(s-t)^2+4abct.$$

The first term is positive or zero, by (20), none of the remaining terms are negative and the last term is positive, hence

$$(\overline{OV}^2 - 4\rho_6^2) > 0.$$

If $t \leq r$, $t \leq s$,

$$\Delta(\overline{OV}^2 - 4\rho_3^2) = \Delta(\overline{OV}^2 - 4\rho_6^2) - 4\Delta(\rho_6^2 - \rho_3^2).$$

The first parenthesis was proved positive in the last paragraph. Combining the last term, 4abct, of the first parenthesis as written in the last paragraph with the second parenthesis get

$$4(s-t)(r-t)(ab-ar-bs+rs+rt+st-t^2)+4ct(ab-rs-t^2+rt+st).$$

The first of these two expressions is positive or zero, the second is positive, hence

$$(\overline{OV}^2 - 4\rho_3^2) > 0.$$

Lemma 3. The lattice point $W_1(x,y,1)$, $x \geq 1$, $y \geq 1$, $xy > 1$, is further from 0 than the lattice point $V(1,1,1)$, hence the plane w_1 does not intersect Π.

$$\overline{OW}_1^2 - \overline{OV}^2 = ax^2+by^2+c+2ry+2sx+2txy-(a+b+c+2r+2s+2t)$$
$$(x^2-1)a+(y^2-1)b+2r(y-1)+2s(x-1)+2t(xy-1).$$

The last expression is greater than zero since $x \geq 1$, $y \geq 1$, $xy > 1$.

Lemma 4. The lattice point $W_2(-x,-y,1)$, $x \geq 2$, $y \geq 2$, is not closer to 0 than the lattice point $V(1,1,1)$, hence w_2 does not intersect Π.

$$\overline{OW}_2^2 - \overline{OV}^2 = ax^2+by^2+c-2ry-2sx+2txy-(a+b+c+2r+2s+2t)$$
$$= (x+1)[(x-1)a-2s]+(y+1)[(y-1)b-2r]-2t(xy-1).$$

Each term of the last expression is positive or zero.

Lemma 5. The lattice point $W_3(-x,y,1)$, $x \geq 2$, $y \geq 2$, $xy > 4$, is not closer to 0 than $V(1,1,1)$, hence w_3 does not intersect Π.

$$\overline{OW}_3^2 - \overline{OV}^2 = ax^2+by^2+c+2ry-2sy-2txy-(a+b+c+2r+2s+2t)$$
$$= (x^2-1)a+(y^2-1)b+2r(y-1)-2s(x+1)-2t(xy+1)$$
$$\geq (x^2+y^2-xy-x-4)a+(y^2-1)(b-a).$$

This expression can not be negative if $(x^2+y^2-xy-x-4)$ is positive or zero. If $y = 2$, $x^2+y^2-xy-x-4 = x^2-3x \geq 0$ if $x \geq 3$. The discriminant of $x^2+y^2-xy-x-4$, considered as a quadratic in x is $-(3y^2-2y-17)$, which is negative if $y \geq 3$, hence $x^2+y^2-xy-x-4$ is positive for all values of x if $y \geq 3$. Hence, the lemma is proved.

Lemma 6. The lattice point $W_4(x,-y,1)$, $x \geq 2$, $y \geq 2$, $xy > 4$, is not closer to 0 than $V(1,1,1)$, hence w_4 does not intersect Π.

$$\overline{OW}_4^2 - \overline{OV}^2 = ax^2+by^2+c-2ry+2sx-2txy-(a+b+c+2r+2s+2t)$$
$$= (x^2-1)a+(y^2-1)b-2r(y-1)+2s(x-1)-2t(xy+1)$$
$$\geq (x^2+y^2-xy-y-4)a+(y^2-y-2)(b-a).$$

This expression can not be negative if $(x^2+y^2-xy-y-4)$ is positive or zero. But this expression is the same as that of lemma 5 with x and y reversed, hence is positive or zero as shown in that lemma.

Let u be any plane known not to cross the interior of Π and u' be any other plane. Let the z axis of the rectangular system intersect u in points whose coordinates are z_u and $z_{u'}$.

Lemma 7. If the projection on the plane POQ of the intersection of u and u' does not cross the interior of hexagon H and if $z_{u'}$ is greater than z_u, then u' does not cross the interior of Π.

Two planes can intersect in but one line. The projection on plane POQ of the intersection of u and u' does not fall within π, hence all points of u' whose projection on plane POQ fall within Π must be either above or below the corresponding points of u. Consequently, since one such point $z_{u'}$ of u' is above the corresponding point z_u of u, the lemma follows.

By analytic geometry, the equation of the plane which is perpendicular bisector of the line joining 0 to the point whose rectangular coordinates are (x_1,y_1,z_1) is $x_1x+y_1y+z_1z = (x_1^2+y_1^2+z_1^2)/2$. The equations of the planes used in the following lemmas are found by use of this and formula (13) of this chapter.

Let h be the altitude on POQ of the tetrahedron OPQM.

Lemma 8. If $r \leq t$, $r \leq s$, plane s' does not intersect the interior of Π, = Π.

The equation of s' is

$$x(a-t+s)/\sqrt{a} - (ar-st+t^2-ab)y/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c-2r+2s-2t)/2.$$

The equation of m is

$$sx/\sqrt{a} - (ar-st)y/\sqrt{a}\sqrt{ab-t^2} + hz = c/2. \tag{24}$$

The equation of the projection on plane POQ of the line of intersection of s' and m is

$$(a-t)x/\sqrt{a} - y\sqrt{(ab-t^2)}/\sqrt{a} = (a+b-2r+2s-2t)/2. \tag{25}$$

This is a line parallel to the side s' of H whose equation is

$$(a-t)x/\sqrt{a} - y\sqrt{(ab-t^2)}/\sqrt{a} = (a+b-2t)/2.$$

Since $r \leq s$, $(a+b-2r+2s-2t) \geq (a+b-2t)$. Hence, (25) lies outside of H if $s > r$ and coincides with the side s' if $s = r$. But

$$z_{s'}-z_m = (a+b-2r+2s-2t)/2h \geq a/2h > 0.$$

By lemma 7, lemma 8 follows.

Lemma 9. If $r \leq t$, $r \leq s$, plane r_1' does not intersect the interior of $\Pi_1 = \Pi$.

The projection on plane POQ of the intersection of r_1' :

$$x(-a-t+s)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c-2r-2s+2t)/2,$$

and p_1' :

$$(-a+s)x/\sqrt{a} + y(ar-st)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+c-2s)/2$$

is

$$-tx/\sqrt{a} - y\sqrt{(ab-t^2)}/\sqrt{a} = (b+2t-2r)/2. \tag{26}$$

This is parallel to side q' of H, whose equation is

$$-tx/\sqrt{a} - y\sqrt{(ab-t^2)}/\sqrt{a} = b/2.$$

Since

$$b+2t-2r \geq b, \text{ for } r \leq t,$$

then (26) is outside of H if $r < t$ and coincides with side q' of H if $r = t$.

Since

$$z_{r_1'}-z_{p_1'} = (b+2t-2r)/2h \geq b/2h > 0,$$

then lemma 9 follows by lemma 7.

Lemma 10. If $s \leq r$, $s \leq t$, plane s_1 does not intersect the interior of $\Pi_2 = \Pi$.

The projection on plane POQ of the intersection of s_1 :

$$x(-a+t+s)/\sqrt{a} + y(ar-st+ab-t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c+2r-2s-2t)/2$$

and m :

$$sx/\sqrt{a} + y(ar-st)/\sqrt{a}\sqrt{ab-t^2} + hz = c/2$$

is

$$x(-a+t)/\sqrt{a} + y\sqrt{(ab-t^2)}/\sqrt{a} = (a+b+2r-2s-2t)/2. \tag{27}$$

This is parallel to side s of H whose equation is

$$x(-a+t)/\sqrt{a} + y\sqrt{(ab-t^2)}/\sqrt{a} = (a+b-2t)/2.$$

Now

$$a+b+2r-2s-2t \geq a+b-2t$$

since $s \leq r$, and therefore (27) is outside of H if $s < r$ and coincides with side s of H if $s = r$.

Since

$$z_{s_1'}-z_m = (a+b-2t-2s+2r)/2h \geq (a+b-2t)/2h > 0,$$

then lemma 10 follows by lemma 7.

Lemma 11. If $s \leq r$, $s \leq t$, r_1' does not intersect the interior of $\Pi_2 = \Pi$.

The projection on plane POQ of the intersection of r_1 :

$$x(-a-t+s)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c-2s-2r+2t)/2$$

and q_1' :

$$x(-t+s)/\sqrt{1} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (b+c-2r)/2$$

is

$$x = -(a+2t-2s)/2\sqrt{a}. \tag{28}$$

This is parallel to side p' of H whose equation is

$$x = -1/2\sqrt{a}.$$

Since $s \leq t$, $a+2t-2s \geq a$, (28) is outside of H if $s < t$ and coincides with side p' of H if $s = t$. Since

$$z_{r_1'} - z_{q_1'} = (a+2t-2s)/2h \geq a/2h > 0,$$

lemma 11 follows by lemma 7.

Lemma 12. If $t \leq r$, $t \leq s$, s_1 does not intersect the interior of $\Pi_3 = \Pi$.

The projection on plane POQ of the intersection of s_1 :

$$x(-a+t+s)/\sqrt{a} + y(ar-st+ab-t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c+2r-2s-2t)/2$$

and p_1' :

$$x(-a+s)/\sqrt{a} + y(ar-st)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+c-2s)/2$$

is

$$tx/\sqrt{a} + y\sqrt{(ab-t^2)}/\sqrt{a} = (b+2r-2t)/2. \tag{29}$$

This is parallel to side q of H whose equation is

$$tx/\sqrt{a} + y\sqrt{(ab-t^2)}/\sqrt{a} = b/2.$$

Since $t \leq r$, $(b+2r-2t) \geq b$, (29) is outside of H if $t < r$ and coincides with side q of H if $t = r$.

Since

$$z_{s_1} - z_{p_1'} = (b+2r-2t)/2h \geq b/2h > 0,$$

lemma 12 follows by lemma 7.

Lemma 13. If $t \leq r$, $t \leq s$, plane s_1' does not intersect the interior of $\Pi_3 = \Pi$.

The projection on plane POQ of the intersection of s_1' :

$$x(-a+s-t)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c-2r+2s-2t)/2$$

and q_1' :

$$x(s-t)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (b+c-2r)/2$$

is

$$x = -(a-2t+2s)/2\sqrt{a}. \tag{30}$$

This is parallel to side p' of H whose equation is

$$x = -\sqrt{a}/2.$$

Now, $a-2t+2s \geq a$ since $t \leq s$. Therefore (30) is outside of H if $t < s$ and coincides with side p' of H if $t = s$.

Since

$$z_{s_1'} - z_{q_1'} = (a-2t+2s)/2h \geq a/2h > 0,$$

lemma 13 follows by lemma 7.

Lemma 14. None of the planes containing faces of Π_1, Π_2, or Π_3 intersect the interior of Π.

This follows from the fact that Π is a convex polyhedron and from lemmas 8-13.

Lemma 15. The plane $w_5(x_1,-1,1)$ does not touch Π if $x_1 \geq 2$.

The projection on the plane POQ of the intersection of w_5 :

$$x(ax_1+s-t)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (ax_1^2+b+c+2r-2sx_1-2tx_1)/2$$

and s_1' :

$$x(a+s-t)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c+2r-2s-2t)/2$$

is

$$x = [a(x_1+1)-2s-2t]/2\sqrt{a}.$$

This is parallel to the side p of H and since $x_1 \geq 2$, is outside of H. Since $x_1 \geq 2$,

$$z_{w_5} - z_{s_1'} = [a(x_1+1)-2s-2t]/2h > 0.$$

Therefore lemma 15 follows by lemma 7.

Lemma 16. The plane $w_6(-x_1,1,1)$ does not touch Π if $x_1 \geq 2$.

This may be shown in a manner similar to that of lemma 15 with s_1 in the place of s_1'.

Lemma 17. The plane $w_7(-1,y_1,1)$ does not touch Π if $y_1 \geq 2$.

The projection on plane POQ of the intersection of w_7 :

$$x(-a+s+ty_1)/\sqrt{a} + y[ar-st+(ab-t^2)y_1]/\sqrt{a}\sqrt{ab-t^2} + hz = (a+by_1^2+c+2ry_1-2s-2ty_1)/2$$

and s_1 :

$$x(-a+s+r)/\sqrt{a} + y(ar-st+ab-t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c+2r-2s-2t)/2$$

is

$$tx + \sqrt{(ab-t^2)}y/\sqrt{a} = [(y_1+1)b+2r-2t]/2.$$

This is parallel to side q of H and, since $y_1 \geq 2$, is outside of H. Since $y_1 \geq 2$,

$$z_{w_7} - z_{s_1} = [(y_1+1)b+2r-2t]/2h > 0.$$

Accordingly, lemma 17 follows by lemma 7.

Lemma 18. The plane $w_8(1,-y_1,1)$ does not touch Π if $y_1 \geq 2$.

This may be shown in a manner similar to lemma 17 with s_1' in place of s_1.

Lemma 19. The plane $w_9(2,-2,1)$ does not touch Π.

The projection on the plane POQ of the intersection of w_9 :

$$x(2a-2t+s)/\sqrt{a} - [ar-st-2(ab-t^2)]\,y/\sqrt{a}\sqrt{ab-t^2} + hz = (4a+4b+c-4r+4s-8t)/2$$

and s_1' :

$$x(a-t+s)/\sqrt{a} + [(ar-st)-(ab-t^2)]\,y/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c-2r+2s-2t)/2$$

is

$$(a-t)/\sqrt{a} - (\sqrt{ab-t^2})y/\sqrt{a} = (3a+3b-2r+2s-6t)/2.$$

This is parallel to the side s' of H whose equation is

$$x(a-t)/\sqrt{a} - y\sqrt{ab-t^2}/\sqrt{a} = (a+b-2t)/2.$$

It follows from

$$2(a-2t)+2(b-r+s) > 0$$

that

$$(3a+3b-2r+2s-6t) > (a+b-2t).$$

Accordingly the projection of the intersection of these two planes is outside of H. Since

$$z_{w_9}-z_{s_1'} = (3a+3b-2r+2s-6t)/2h > 0,$$

lemma 19 follows by lemma 7.

Lemma 20. The plane $w_{10}(-2,2,1)$ does not touch Π.

This may be shown in a manner similar to that of lemma 19 with s_1 in place of s_1'.

Lemmas 1-20 show that the plane which is the perpendicular bisector of the line joining O to any lattice point of the plane POQ or the plane MP_1Q_1 does not cross the interior of Π.

Lemma 21. The plane $w_{11}(-x_1,-1,1)$, $x_1 \geq 2$, does not touch Π.

The projection on the plane POQ of the intersection of w_{11} :

$$x(-ax_1+s-t)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (ax_1^2+b+c+2x_1t-2r-2x_1s)/2$$

and r_1' :

$$x(-a-s-t)/\sqrt{a} + y(ar-st-ab+t^2)/\sqrt{a}\sqrt{ab-t^2} + hz = (a+b+c+2t-2r-2s)/2$$

is

$$x = [(x_1+1)a+2t-2s]/2\sqrt{a}.$$

This is parallel to the side p' of H whose equation is

$$x = -1/2\sqrt{a}.$$

Obviously the projection of the intersection of these two planes is outside of H. Since

$$z_{w_{11}} - z_{r_1'} = [(x_1-1)][a(x_1+1)-2t-2s]/2h > 0,$$

lemma 21 follows by lemma 7.

Lemma 22. The plane $w_{12}(-1,-y_1,1)$, $y_1 \geq 2$, does not touch Π.

This may be shown in a manner similar to that of lemma 21 with q' instead of p'.

Lemma 23. The perpendicular distance to the second plane above POQ is greater than twice the distance from O to the farthest vertex of Π.

Case 1. $r \leq s$, $r \leq t$, hence $\pi = \pi_1$.

It must be shown that $4h^2 > 4\rho_4^2$. We may write

$$4\Delta(ab-t^2)(h^2-\rho_4^2)$$

as

$$(abc-ct^2-2ar^2-2bs^2+4rst)^2+(ab-t^2)[(c-a)(abc-2brs-2crt) +2c(t-r)(ab-bs-ct)+r^2(a-2t-2s+2r)^2-4r(t+s-r)(bc-bs -ct)+(c-b)abc+8rs(t-r)c+2c(a-2s)(bs-2r^2)+b^2s^2].$$

Every term of this expression is either positive or zero and $(ab-t^2)b^2s^2$ is necessarily positive, hence $h^2 > \rho_4^2$ and lemma 21 holds for case 1.

Case 2. $s \leq t$, $s \leq r$, hence $\pi = \pi_2$.

In this case we must prove that $h^2 > \rho_5^2$. Now

$$4\Delta(ab-t^2)(h^2-\rho_5^2)$$

is equal to

$$(abc-ct^2-2ar^2-2bs^2+4rst)^2+(ab-t^2)[(c-a)abc+2b(t-s)(ac-ar -ct)+(c-b)(abc-2art-2ct^2)+s^2(b-2r-2t+2s)^2+4s(r+t-s) (ac-ar-ct)+a^2r^2+2c(b-2r)(ar-2s^2)+8rs(t-s)c].$$

Every term of this expression is likewise positive or zero and

$$(ab-t^2)a^2r^2$$

is positive. Therefore $h^2 > \rho_5^2$.

Case 3. $t \leq s$, $t \leq r$, hence $\pi = \pi_3$.

In this case we must show that $h^2 > \rho_6^2$, and $h^2 > \rho_3^2$, since either ρ_3 or ρ_6 may be the greater. Now

$$4\Delta(ab-t^2)(h^2-\rho_6^2)$$

is equal to

$$(abc-ct-2ar-2bs+4rst)^2+(ab-t^2)[c(2c-a-b)(ab-2t^2)+2c(a-2s)(bs-2rt)+2ac(r-t)(b-2r)+(ar-bs)^2+t^2(c-2r-2s+2t)^2+4t(r+s-t)(ab-ar-bs)+2bct(s-t)+2act(r-t)+c^2t^2].$$

Every term of this expression is positive or zero and

$$(ab-t^2)c^2t^2$$

is positive, hence

$$h^2 > \rho_6^2.$$

To prove that $h^2 > \rho_3^2$, note that

$$4\Delta(ab-t^2)(h^2-\rho_3^2)$$

is equal to

$$(abc-ct^2-2ar^2-2bs^2+4rst)^2+(ab-t^2)[2c(a-2s)(bs-2rt)+2ac(r-t)(b-2r)+(ar+bs-2rs)^2+2crt(a-2s)+2bct(s-t)+(c-b)c(ab-2t^2)].$$

Every term of this is positive or zero and the first parenthesis is positive and not less than c^2t^4.

This concludes the proof of lemma 21 and also of theorem 12.

Let ρ be the distance of the farthest vertex of π from 0.

Theorem 13. $4\Delta(c-\rho^2) \geq abc^2$.

Case 1. $r \leq t$, $r \leq s$, $\rho = \rho_4$.

(a) If $t \geq s \geq r$, write $4\Delta(c-\rho_4^2)-abc^2$ as

$$2c[(bt-2r^2)(a-2t)+4rt(s-r)]+r^2(a-2s-2t+2r)^2+4r(s+t-r)(bc-bs-ct)+(c-b)(abc-3ct^2-bs^2+2crt)+(c-a)(abc-2brs-2crt)+2bc(s-r)(a-s-t)+bc(t^2-s^2) \geq 0,$$

since each term is positive or zero.

(b) If $s \geq t$, write $4\Delta(c-\rho_4^2)-abc^2$ as

$$2c[(bs-2r^2)(a-2s)+4rs(t-r)]+r^2(a-2s-2t+2r)^2+4r(s+t-r)(bc-bs-ct)+bc(s^2-t^2)+(c-a)(abc-2brs-2ct^2)+(c-b)(abc-bs^2-ct^2)+ac(t-r)(b-2t)+bc(t-r)(a-2s) \geq 0$$

since each term is positive or zero.

Case 2. $s \leq t$, $s \leq r$, $\rho = \rho_5$.

(a) If $t \geq r$, write $4\Delta(c-\rho_5^2)-abc^2$ as

$$2c[(bt-2r^2)(a-2t)]+2c(ab-4rt)(r-s)+s^2(b-2r-2t+2s)^2+4s(r+t-s)(ac-ar-ct)+(ar-ct)^2+c(c-b)(ab-4t^2)+2bcs(r-s)+b(c-a)(ac-2rs)+2bcs(t-s).$$

(b) If $r > t$, write $4\Delta(c-\rho_5^2)-abc^2$ as

$$2c[(ar-2s^2)(b-2r)+4rs(t-s)]+s^2(b-2r-2t+2s)^2+4s(r+t-s)(ac-ar-ct)+(c-b)(abc-2ct^2-2ars)+(c-a)(abc-ar^2-ct^2)+2(t-s)(abc-bct-acr)+(r-t)(act+acr).$$

Each term of the expressions (a) and (b) is positive or zero under the hypothesis stated. Hence,

$$4\Delta(c-\rho_5^2)-abc^2 \geq 0.$$

Case 3. $t \leq s$, $t \leq r$, ρ = the greater of ρ_4 or ρ_3.

(a) If $r \geq s$, write $4\Delta(c-\rho_6^2)-abc^2$ as

$$2c[(b-2r)(ar-2t^2)+4rt(s-t)]+(ar-bs)^2+t^2(c-2r-2s+2t)^2+2bc(a-2s)(s-t)+4t(r+s-t)(ab-ar-bs)+c(c-b)(ab-4t^2+2rt)+c(c-a)(ab-2rt)+2bct(r-s) \geq 0,$$

since each term is positive or zero.

(b) If $s > r$, write $4\Delta(c-\rho_6^2)-abc^2$ as

$$2c[(bs-2t^2)(a-2s)+4st(r-t)]+c(c-a)(ab-4t^2)+4t(s+r-t)(ab-ar-bs)+t^2(c-2s-2r+2t)^2+(bs-ar)^2+2ct(bs-ar)+2ac(r-t)(b-2r)+abc(c-b) \geq 0,$$

since each term is positive or zero.

(c) If $r \geq s$, write $4\Delta(c-\rho_3^2)-abc^2$ as

$$2c[(ar-2s^2)(b-2r)]+2(s-t)(abc-4crs)+2(c-b)(abc-2bt^2-2ct^2)+[ar+bs+(b+c)t-2rs][ar+bs+(c-b)t-2rs]+(b-a)abc \geq 0,$$

since each term is positive or zero.

(d) If $s > r$, write $4\Delta(c-\rho_3^2)-abc^2$ as

$$2c[(bs-2r^2)(a-2s)]+2(r-t)(abc-4crs)+2(c-b)(abc-2bt^2-2ct^2)+[ar+bs+t(b+c)-2rs][ar+bs+(c-b)t-2rs]+(b-a)abc \geq 0,$$

since each term is positive or zero.

This completes the proof of theorem 13.

Theorem 14. $3c/4 > \rho^2$.

Write $\Delta(3c-4\rho^2)$ as

$$4\Delta(c-\rho^2)-abc^2+c(ar^2-2rst+bs^2+ct^2).$$

The first term is positive or zero by theorem 13, the last is necessarily positive. Theorem 14 follows.

Case II. $r \leq 0$, $s \leq 0$, $t \leq 0$.

In this case there is only one polyhedron Π_4 whose interior is the locus of all points closer to 0 than to any other lattice point. Its vertices lying above the plane POQ are now listed in order according to the faces. As in case I, following each vertex is listed its distance from 0. Throughout case II, $\pi = \pi_4$.

Π_4

Face m	Face r_1	Face q_1	Face p_1
$m\ p_1\ r_1\ \rho_{-1}$	$r_1\ p_1\ m\ \rho_{-1}$	$q_1\ m\ p'\ \rho_{-3}$	$p_1\ m\ r_1\ \rho_{-1}$
$m\ r_1\ q_1\ \rho_{-2}$	$r_1\ m\ q_1\ \rho_{-2}$	$q_1\ p'q\ \rho_{-2}$	$p_1\ r_1\ p\ \rho_{-3}$
$m\ q_1\ p'\ \rho_{-3}$	$r_1\ q_1\ q\ \rho_{-3}$	$q_1\ q\ r_1\ \rho_{-3}$	$p_1\ p\ q'\ \rho_{-1}$
$m\ p'r'\ \rho_{-1}$	$r_1\ q\ r\ \rho_{-1}$	$q_1\ r_1\ m\ \rho_{-2}$	$p_1\ q'm\ \rho_{-3}$
$m\ r'q'\ \rho_{-2}$	$r_1\ r\ p\ \rho_{-2}$		
$m\ q'p_1\ \rho_{-3}$	$r_1\ p\ p_1\ \rho_{-3}$		

Under certain conditions some faces reduce to edges or vertices. This will not affect the validity of the proofs which follow so these cases will not be considered separately.

Using (22) get

$$4\Delta\rho_{-i} = abc(a+b+c+2r+2s+2t)-E_i, \text{ where } i = 1,2,3$$

and

$$E_1 = (-ar+bs+ct+2st)^2,$$
$$E_2 = (ar-bs+ct+2rt)^2,$$
$$E_3 = (ar+bs-ct+2rs)^2.$$

Theorem 15. The interior of Π_4 is the locus of all points closer to 0 than to any other lattice point.

As in case I, no point outside of Π_4 is closer to 0 than to any other lattice point. It remains to prove that no plane which is the perpendicular bisector of the line joining 0 to some other lattice point crosses the interior of Π_4. Proof of this will be made by a series of lemmas.

Lemma 1. The distance of the point w(2,2,1) from 0 is more than twice the distance of 0 from the farthest vertex of Π.

Write $\Delta(\overline{OW}^2-4\rho_{-1}^2)$ as

$$(c-b)(ab-4t^2)(a+b+2t+2r+2s)+(2rst-ar^2-bs^2-bt^2+ab^2/2)$$
$$+2st(b+2t)+t^2(b+2r)+(ab+4rt^2)/2+2ab(a+2t)+(ab$$
$$+4st^2-2art)(bs+bt-ar+2st)^2+(ab^2+4rst-2ar^2-2bs^2$$
$$-2bt^2)(a+2b+2t+2r+2s)+(2a+b+4t)(ab^2+2rst-ar^2$$
$$-bs^2-bt^2).$$

Each term of this is positive or zero, in particular the last is necessarily positive. Hence,

$$(\overline{OW}^2-4\rho_{-1}^2) > 0.$$

Similarly, write $\Delta(OW^2-4\rho_{-2}^2)$ as

$$(c-b)[(ab-4t^2)(a+b+2r+2s+2t)+(2rst-ar^2-bs^2-bt^2+ab^2/2)$$
$$+2ab(a+2t)+(ab-4st)/2b+2rt(a+2t)+(ab^2+4st^2+4rt^2)]$$
$$-(ab^2+4rst-2ar^2-2bs^2-2bt^2)(a+2b+2t+2r+2s)+(2a+b

$$+4t)(ab^2+2rst+ar^2-bs^2-bt^2)+(ar+bt+2rt-bs)^2 > 0.$$

Each term is positive or zero and the last is positive. Hence

$$\overline{OW}^2 > 4\rho_2^2.$$

Finally, write $\Delta(\overline{OW}^2-4\rho_3^2)$ as

$$(c-b)[(ab-4t^2)(a+b+2t+2r+2s)+(2rst-ar^2-bs^2-bt^2+ab^2/2)$$
$$+2ab(a+2t)-ar(b+2t)+(b+2r)(ab-4st)/2+(ab^2+4st^2$$
$$+2rt^2)+t^2(b+2r)]-(ar+bs-bt+2rs)^2+(ab+4rst-2ar^2$$
$$-2bs^2-2bt^2)(a+2b+2r+2s+2t)+(2a+b+4t)(ab^2+2rst$$
$$-ar^2-bs^2-bt^2) > 0.$$

Each term is positive or zero and the last is positive. Hence

$$\overline{OW}^2 > 4\rho_3^2.$$

Lemma 2. The distance from 0 to any point $T(x,y,1)$, $|x| \geq 2$, $|y| \geq 2$, is greater than or equal to that of $w(2,2,1)$.

If $x \geq y \geq 2$, then $\overline{OT}^2 - \overline{OW}^2$ may be written

$$(x-2)[a(x+2)+2ty+2s]+(y-2)[b(y+2)+4t+2r].$$

Each bracket is positive and has a positive or zero coefficient. Accordingly, the lemma is true for this case.

If $y \geq x \geq 2$, then $\overline{OT}^2-\overline{OW}^2$ may be written

$$(x-2)[a(x+2)+4t+2s]+(y-2)[b(y+2)+2tx+2r],$$

which is positive or zero since each bracket is positive and has a positive or zero coefficient. Therefore the entire expression is not less than zero and the lemma is true if neither x nor y is less than two.

If x and y are not both positive, let

$$x' = |x|, \quad y' = |y|$$

and consider $T'(x',y',1)$ and $T(x,y,1)$. Write $\overline{OT}^2-\overline{OT'}^2$ as

$$2[t(xy-x'y')+(x-x')s+(y-y')r].$$

Since each parenthesis is negative or zero and each coefficient r,s and t is likewise negative or zero, the entire expression is positive or zero, so

$$\overline{OT}^2 \geq \overline{OT'}^2 \geq \overline{OW}^2.$$

Lemma 1 of theorem 12 and its proof are as valid when r,s and t are zero or negative as when r,s and t are positive. Hexagon H of lemma 12 has the sides p,r,q,p',r' and q' when r,s and t are negative or zero.

Lemma 3. The plane which is the perpendicular bisector of the line joining 0 to the point having the coordinates $(x_1,y_1,1)$ with at least one of $|x_1|$ and $|y_1|$ equal to 1 but x_1 and y_1 not both 1, $|x_1| \geq 1$, $|y_1| \geq 1$, does not intersect π.

The proof of this lemma is analogous to those of lemmas 8-13 of theorem 12.

Lemma 4. The second lattice plane above the plane POQ is at a greater distance from 0 than $2\rho_1$, $2\rho_2$, or $2\rho_3$.

Write $4\Delta(ab-t^2)(h^2-\rho_i^2)$ as

$$(32)\qquad (abc+4rst-2ar^2-2bs^2-ct^2)^2+(ab-t^2)\{E_1+c[ct^2-2ar(b+2r)-2bs(a+2s)-2bt(a+2t)+8rst+(c-b)(ab-4t^2)+(c-a)ab]\}.$$

To prove this expression positive or zero, consider two cases.

(a) Suppose $4a \geq b$.

Let $F=-2br(b+2r)-2bs(a+2s)-2bt(a+2t)+8rst$. Then (32) may be written

$$(33)\qquad (abc-ct^2-2ar^2+4rst-2bs^2)^2+(ab-t^2)\{E_1-c[ct^2+(c-b)(2ab-4t^2+[(b-a)/4](\overline{4r-b}^2+b.\overline{4a-b})+F]\}.$$

Every term of (33) is positive or zero by inspection except F. To prove F is positive or zero, let u,v and w be r,s and t in some order and let u be the smallest in absolute value. Then

$$F \geq [-2bu(b+2r)-2bu(a+2s)-2bu(a+2t)+8uvw]= -2bu[(a+b+2r+2s+2t)-2u(ab-4vw)] \geq 0.$$

Since each parenthesis of this last expression is positive or zero, this proves $F \geq 0$. By (20) of this chapter

$$(abc-ct^2-2ar^2-2bs^2-4rst)^2 \geq c^2t^4.$$

This is the first parenthesis of (32), so (32) is positive if $t \neq 0$. If $t=0$, then

$$(abc-ct^2-2ar^2-2bs^2+4rst)^2= (abc-2ar^2-2bs^2)^2.$$

This last parenthesis is positive unless $a=b=c=-2s=-2r$.

(b) Suppose $b > 4a$.

Write $4\Delta(ab-t^2)(h^2-\rho_i^2)$ as

$$(abc+4rst-2ar^2-2bs^2-ct^2)^2+(ab-t^2)\{E_1+c[ct^2-2ar(b+2r)-2bs(a+2s)-2bt(a+2t)+(c-b)(2ab-4t^2)+(\overline{b-a}.ab+8rst)]\}.$$

Each term is positive or zero and the last parenthesis, $[(b-a)ab+8rst]$ is positive since $(b-a)ab > 3ab^2/4$ and $-8rst > a^2b > ab^2/4$. These inequalities follow from the fact that $b > 4a$.

If $2\rho_1 = 2h$, the plane MP_1Q_1 which is the perpendicular bisector of the perpendicular from 0 to the second lattice plane above POQ, is tangent to the sphere with radius ρ_1 about 0 as the center, so it does not intersect Π. Therefore no perpendicular bisector of the line joining 0 to any point in the second or higher lattice planes above POQ intersects Π. This completes the

proof of theorem 15.

Theorem 16. $4\Delta(c-\rho_i^2)-abc^2 \geq 0$, $i = 1,2,3$.

(a). If $4a \geq b$, the expression $4\Delta(c-\rho_i^2)-abc^2$ may be written

$$2c(c-b)(ab-2t^2)+E_1-[c(b-a)/4]\,[b(4a-b)+(b+2r)^2]+c\,[-2br(b+2r) -2bs(a+2s)-bt(a+2t)+8rst].$$

The expression in the last bracket is defined as F in lemma 4 of theorem 15. It was proved there that $F \geq 0$. Each of the other parentheses is positive or zero. Hence $4\Delta(c-\rho_i^2)-abc^2 \geq 0$.

(b). If $b > 4a$, write $4\Delta(c-\rho_i^2)-abc^2$ as

$$c(c-b)(ab-4t^2)+E_1+2c[-ar(b+2r)-bt(a+2t)-bs(a+2s)+(ab/4) (c-4a)]+(c/4)(3abc+16rst).$$

Each of these parentheses is positive or zero. Accordingly,

$$4\Delta(c-\rho_i^2)-abc^2 \geq 0.$$

Theorem 17. $3c/4 \geq \rho_i^2$, $i = 1,2,3$.

Write $\Delta(3c-4\rho_i^2)$ as

$$[4\Delta(c-\rho_i^2)-abc^2]+c\,[(ar^2+2rst+bs^2)+ct^2].$$

The first bracket is positive or zero as shown in theorem 16. The second is positive unless $r = s = t = 0$, in which case it is zero. If $r = s = t = 0$, then $\Delta(3c-4\rho_i^2) = ab(2c-a-b)$, which is positive unless $a = b = c$. Therefore,

$$3c/4 > \rho_i^2,$$

unless $r = s = t = 0$, and $a = b = c$, in which case

$$3c/4 = \rho_i^2.$$

CHAPTER III

QUATERNARY FORMS; FOUR DIMENSIONAL POINT LATTICES

Definition. A fundamental hyper-parallelopiped is bounded by 8 parallelopipeds whose vertices are lattice points. These bounding parallelopipeds are called cells.

Start with any point lattice in four dimensions. Locate lattice points O,P,Q,M as described in chapter ii. Next, choose a lattice point N not lying in the hyperplane determined by O,P, Q,M as close to O as possible. The hyper-parallelopiped

$$F = \{P,Q,M,N\}$$

having the edges OP, OQ, OM, ON is called a reduced fundamental hyper-parallelopiped of the lattice L. It has the following properties:

(i) No edge exceeds a diagonal.

(ii) No edge of a cell exceeds a diagonal of it.

(iii) No edge of a face of a cell exceeds a diagonal of that face.

(iv) $OP \leq OQ \leq OM \leq ON$.

By Dickson, (ii), (iii) and (iv) hold for the cell determined by (O,P,Q,M) and hence for the upper cell opposite it. Each remaining diagonal is either a line joining O to a point of the hyperplane of the upper cell U or is parallel and equal to such a line. Therefore, it is greater than or equal to ON.

Conversely, if F is any fundamental hyper-parallelopiped (P,Q,M,N) of any point lattice L of four dimensional space and if F has properties (i)-(iv), then F is reduced for L. By Dickson, OP is the minimum distance from O to any lattice point in the plane POQ, while OQ is the minimum distance from O to lattice points in that plane but not on line OP, and OM is the minimum distance from O to lattice points in the hyperplane OPQM but not in the plane POQ.

Consider the lattice points in the hyperplane U of the upper cell of F. By (i), (ii) and (iii), ON is not greater than any of the lines joining O to the 26 lattice points of U which are vertices of the eight parallelopipeds with the common vertex

N and one of which is the upper cell of F. Hence the foot T of the perpendicular from O to the hyperplane U is not further from N than from any of those 26 points. Hence T lies on or within the polyhedron Π. By theorems 12 and 13, the distance from any lattice point in U from T is not less than NT, whence its distance from O is not less than ON. If ρ is the distance of the farthest vertex of polyhedron Π from N so that

$$NT \leq \rho,$$

evidently

$$\overline{OT}^2 \geq \overline{ON}^2 - \rho^2. \tag{34}$$

By theorems 14 and 17,

$$\rho^2 \leq 3\overline{OM}^2/4 \leq 3\overline{ON}^2/4.$$

Hence, $OT \geq ON/2$ and the distance from O to the second parallel hyperplane is not greater than 2ON. This proves the converse.

Define a,b,c,r,s and t as in (12) of chapter ii. Let

$$d = \overline{ON}^2,$$
$$i = \sqrt{ad}\cos PON,$$
$$j = \sqrt{bd}\cos QON,$$
$$k = \sqrt{cd}\cos MON.$$

Then

$$a \leq b \leq c \leq d.$$

Consider OP, OQ, OM, ON as vectors. Letx, y, z, w represent any integers. Then the vector from O to any lattice point V of L is

$$x\overline{OP}+y\overline{OQ}+z\overline{OM}+w\overline{ON}.$$

As in chapter ii, let (x,y,z,w) be the oblique coordinates of the point V. The square of the distance from O to V is given by

$$\overline{OV}^2 = ax^2+by^2+cz^2+dw^2+2ryz+2sxz+2txy+2ixw+2jyw+2kzw = \phi.$$

The coordinates of P, Q, M and N in the rectangular system used in chapter i are given by (2) with $P_1 = P$, $P_2 = Q$, $P_3 = M$ and $P_4 = N$ and a_{ij} is the element in the i^{th} row j^{th} column of the determinant

$$\delta = \begin{vmatrix} a & t & s & i \\ t & b & r & j \\ s & r & c & k \\ i & j & k & d \end{vmatrix}.$$

Let us now examine the case in which $OT = ON/2$.

By theorem 17,

$$OP \leq OQ \leq OM \leq ON$$

and the first three of these lines are mutually perpendicular. Let T_1 be the foot of the perpendicular from O to the second

lattice hyperplane U_1 above OPQM. If T_1 is not a lattice point, then ON is less than the distance from O to any lattice point in U_1. The first three rectangular coordinates of any lattice point in U_1 whose oblique coordinates are (x,y,z,2) are, noting that $r = s = t = 0$, :

(35) $$(xa+21)/\sqrt{a},\ (yb+2j)/\sqrt{b},\ (zc+2k)/\sqrt{c}.$$

These coordinates must be zero if the lattice point is to be T_1. In the cell OPQN, the squares of the diagonals of the faces are given by :

$$a \pm 2i+d,\ b \pm 2j+d,\ c \pm 2k+d.$$

By (iii), each of these expressions must be greater than or equal to d. Accordingly,

$$a-2|i| = 0,\ b-2|j| = 0,\ c-2|k| = 0.$$

So, if the coordinates given in (35) are to be zero,

$$x^2 = y^2 = z^2 = 1$$

and

$$a-2|i| = b-2|j| = c-2|k| = 0.$$

We conclude that if a lattice point T_1 in U_1 is at the same distance from O that N is, then

$$a = b = c = d = 2|i| = 2|j| = 2|k|,$$

and that OT_1 is the perpendicular from O to U_1.

Therefore, if T_1 is a lattice point in U_1 and $OT_1 = ON$, then

$$a = b = c = d = 2|i| = 2|j| = 2|k|,$$

and

$$r = s = t = 0.$$

Suppose that this is true, and that i, j and k are positive. Then the form ϕ may be replaced by an equivalent form Φ, by means of the transformation

$$x = w_1,\ y = -y_1,\ z = z_1,\ w = x_1+y_1.$$

Now Φ has $A = B = C = D = 2R = 2S = 2T = 2I = 2J$, and $K = 0$. If i, j and k are not all positive, then, with slight adjustment of signs in the transformation, ϕ may still be replaced by Φ.

Thus, we have proved that in every case a reduced fundamental hyper-parallelopiped may be chosen so that every lattice point in the second parallel hyperplane above OPQM is at a greater distance from O than N is.

Theorem 18. Every positive quaternary form of determinant δ is equivalent to one in which $abcd \leq 4\delta$.

By (5), $\sqrt{\delta}$ is the hypervolume of F and thus is the product of its altitude OT by its base. By (5), the square of the volume of its base is

$$\begin{vmatrix} a & t & s \\ t & b & r \\ s & r & c \end{vmatrix}.$$

By (34),

$$\overline{OT}^2 \geq d - \rho^2.$$

Let $d = c-u$. Theorem 18 is true if

$$4\Delta(d-\rho^2) \geq abcd,$$

and hence if

$$[4\Delta(c-\rho^2)-abc^2]+u(4\Delta-abc) \geq 0.$$

By theorems 13 and 16,

$$4\Delta(c-\rho^2)-abc^2 \geq 0$$

and

$$4\Delta-abc \geq abc > 0,$$

by (20). Therefore theorem 18 is true.

Since $a \leq b \leq c \leq d$, it follows from theorem 18 that

$$a^4 \leq 4\delta.$$

But a is the minimum of ϕ.

Corollary. The minimum of every positive quaternary quadratic form of determinant δ is $\leq g = \sqrt[4]{4\delta}$. Also, g is actually the minimum of $g\psi$ of determinant δ where $\psi = x^2+y^2+z^2+w^2+xw+yw+zw$ has the minimum 1 and the determinant 1/4.

CHAPTER IV

REDUCED POSITIVE QUATERNARY QUADRATIC FORMS

In chapter iii it was shown that any form is properly equivalent to a form

$$ax^2+by^2+cz^2+dw^2+2ryz+2sxz+2txy+2ixw+2jyw+2kzw$$

whose hyper-parallelopiped is reduced. If $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$, from the definition of a reduced hyper-parallelopiped the following conditions are satisfied :

(1). $a \le b \le c \le d$.

(2). $a+b+c+2\sigma_2\sigma_3 r+2\sigma_1\sigma_3 s+2\sigma_1\sigma_2 t+2\sigma_1 i+2\sigma_2 j+2\sigma_3 k \ge 0$.

(3). $a+b+2\sigma_2 r+2\sigma_1 s+2\sigma_1\sigma_2 t \ge 0$.
$a+b+2\sigma_1 i+2\sigma_2 j+2\sigma_1\sigma_2 t \ge 0$.
$a+c+2\sigma_1 i+2\sigma_3 k+2\sigma_1\sigma_3 s \ge 0$.
$b+c+2\sigma_2 j+2\sigma_3 k+2\sigma_2\sigma_3 r \ge 0$.

(4). $a-2|t| \ge 0$, $a-2|s| \ge 0$, $a-2|i| \ge 0$, $b-2|r| \ge 0$, $b-2|j| \ge 0$, $c-2|k| \ge 0$.

Condition (1) arises from the manner in which the points P, Q, M and N were chosen, (2) from the fact that every diagonal of the hyper-parallelopiped must be not less than the longest edge,d, of it, (3) from the fact that every diagonal of a cell must be not less than the longest edge of it, and (4) from the fact that every diagonal of a face of a cell must be not less than the face's greater edge.

Definition. $\sigma = 1$, if $rst > 0$, $\sigma = -1$, if $rst \le 0$.

Mr. Dickson has shown that the following conditions may be satisfied by the proper choice of the parallelopiped P,Q,M :

(5). $\sigma = 1$: $|s| \le 2|r|$, if $a = 2|t|$; $|t| \le 2|r|$, if $a = 2|s|$; $|t| \le 2|s|$, if $b = 2|r|$.

(6). $\sigma = -1$: $s = 0$, if $a-2|t| = 0$; $t = 0$, if $a-2|s| = 0$; $t = 0$, if $b-2|r| = 0$. If $a+b+2\sigma_2 r-2\sigma_1 s-2\sigma_1\sigma_2 t = 0$, then $\sigma = -1$, and $a+2\sigma_1 s-\sigma_1\sigma_2 t \le 0$.

(7). $|r| \le |s|$, if $a = b$, $|s| \le |t|$, if $b = c$.

(7a). $|r| \le |s| \le |t|$, if $a = b = c$.

In the following section will be developed conditions imposed on ϕ. We shall start with the assumption that conditions

(1)-(7a) are satisfied.

Definition. $v = 1$ if $i \geq 0$, $j \geq 0$, $k \geq 0$; $v = -1$ if at least one of these coefficients is not zero and $i \leq 0$, $j \leq 0$, $k \leq 0$.

Notation. $\overset{=}{>}$ means that equality may hold only if $v = 1$. $\geq$ means that equality may hold only if $v = -1$. $\geq$ means that equality may hold regardless of the sign of v.

(8). $i \geq 0$, $j \geq 0$, $k \geq 0$ or $i \leq 0$, $j \leq 0$, $k \leq 0$.

(a) If $i = 0$, then $t \geq 0$ and if $t = 0$, then $s \overset{=}{>} 0$.

(b) If $j = 0$, then $t \geq 0$ and if $t = 0$, then $r \overset{=}{>} 0$.

(c) If $k = 0$, then $s \geq 0$ and if $s = 0$, then $r \overset{=}{>} 0$.

(d) If $i = j = 0$, then $s \geq 0$ and if $s = 0$, then $r \geq 0$.

(9). If $a - 2vi = 0$, then $2vj \geq t \geq 0$.

(a) If $t = 0$ or if $2vj = t$, then $2vk \geq s \geq 0$.

(b) If $t = 0$, $s = 2vk$ or if $t = 2vj$, $s = 0$, then $r \geq 0$.

(c) If $s = 2vk$, $t = 2vj$ or if $s = t = 0$, then $v = 1$.

(10). $2vi \geq t \geq 0$ if $b - 2vj = 0$.

(a) $2vk \geq r \geq 0$ if $t = 0$ or if $2vi = t$.

(b) $s \geq 0$ if $t = 0$, $r = 2vk$ or if $r = 0$, $t = 2vi$.

(c) $v = 1$ if $2vi = t$, $r = 2vk$ or if $t = r = 0$.

(11). $a + t - 2vi \leq 0$ if $a + b + 2t - 2vi - 2vj = 0$.

(a) $2vk \overset{=}{>} r + s \geq 0$ if $a + t - 2vi = 0$.

(b) $s \geq 0$ if $s + r = 0$, $a + t - 2vi = 0$.

(12). $2vi \geq s \geq 0$ if $c - 2vk = 0$.

(a) $2vj \geq r \geq 0$ if $2vi = s$ or if $s = 0$.

(b) $t \geq 0$ if $s = 0$ and $2vj = r$ or if $s = 2vi$ and $r = 0$.

(c) $v = 1$ if $s = 2vi$ and $r = 2vj$ or if $s = r = 0$.

(13). $a + s - 2vi \leq 0$ if $a + c + 2s - 2vi - 2vk = 0$.

(a) $2vj \overset{=}{>} t + r \geq 0$ if $a + s - 2vi = 0$.

(b) $t \geq 0$ if $a + s - 2vi = 0$ and $t + r = 0$.

(14). $2vi \geq s + t \geq 0$ if $b + c + 2r - 2vj - 2vk = 0$.

(a) $b + r - 2vj \leq 0$ if $2vi = s + t$ or if $s + t = 0$.

(b) $v = 1$ if $2vi = s + t$ and $b + r - 2vj = 0$.

(c) $t \geq 0$ if $t + s = 0$ and $b + r - 2vj = 0$.

(15). $a + t + s - 2vi \leq 0$ if $a + b + c + 2r + 2s + 2t - 2vi - 2vj - 2vk = 0$.

(a) $b + r + t - 2vj \overset{=}{<} 0$ if $a + t + s - 2vi = 0$.

(16). $a - t - s = 0$ if $a + b + c + 2r - 2s - 2t + 2vi - 2vj - 2vk = 0$ and $b + r - 2vj - s + vi \leq 0$.

(17). $a - t + s - 2vi < 0$ if $a + b + c - 2t + 2s - 2r - 2vi + 2vj - 2vk = 0$.

(18). $a + t - s - 2vi \leq 0$ if $a + b + c + 2t - 2s - 2r - 2vi - 2vj + 2vk = 0$.

(19). $2vi > s > t \geq 0$ if $b + c - 2r + 2vj - 2vk = 0$.

(20). $t > 0$ and $2vj \geq vi$ if $a-|t|-|s|=0$.

(a) $s > 0$ and $2vk \geq vi$ if $2vj = vi$.

(b) $a+s-2vi < 0$ if $a+c+2s-2vi-2vk = 0$.

(c) $v = 1$ if $a-2vi = 0$ or if $i = 0$.

(d) $b+r+vi-2vj \leq 0$ if $b+c+2r-2vj-2vk = 0$.

(21). (I) $t > 0$ and $2vj \neq vi$ if $a-2|t|=0$ and $2|r|=|s| > 0$.

(a) $2vk \geq s > 0$ if $a-2vi = 0$.

(b) $s > 0$ if $i = 0$.

(II) $t > 0$ and $2vj \geq vi$ if $a-2|t|=0$ and $s = 0$.

(a) $r \neq 0$ if $2j = i$.

(b) $v = 1$ if $a-2vi = 0$ or if $i = 0$.

(c) $2vj \geq vi+r$ if $c-2vk = 0$.

(22). (I) $s > 0$ and $2vk \geq vi$ if $a-2|s|=0$ and $2|r|=|t| > 0$.

(II) $s > 0$ and $2vk \geq vi$ if $a-2|s|=0$ and $t = 0$.

(a) $r \neq 0$ if $2vk = vi$.

(b) $v = 1$ if $a-2vi = 0$ or if $i = 0$.

(c) $2vk \geq vi+r$ if $b-2vj = 0$.

(23). (I) $r > 0$ and $2vk \neq vj$ if $b-2|r|=0$ and $2|s|=|t| > 0$.

(II) $r > 0$ and $2vk \geq vj$ if $b-2|r|=0$ and $t = 0$.

(a) $s \neq 0$ if $2k = j$.

(b) $2vk \geq vj+s$ if $a-2vi = 0$.

(c) $v = 1$ if $b-2vj = 0$ or if $j = 0$.

(24). $a+t+2s < 0$ if $a+b+2t+2s+2r = 0$.

(25). $2vk \neq vi+vj$ if $a+b+2t-2s-2r = 0$ and $a+t-2s = 0$.

(26). $2vk+vi \neq vj > vi$ if $a+b+2s-2r-2t = 0$ and $a+2s-t = 0$.

(27). $2vk+vj \neq vi \geq vj$ if $a+b+2r-2s-2t = 0$ and $a-2s-t = 0$.

(28). $vj \geq vk$ if $b = c$ and $|t|=|s|$.

(a) $t \geq 0$ if $j = k$ and $t+s = 0$. And $v = 1$ if $t = s$.

(b) $vj+vk \geq t \geq 0$ if $a-2vi = 0$ and $vj \geq vk+t$ if $s < 0$.

(c) $vj+vk \neq vi$ if $a-t-|s|=0$.

(1) $vj \neq vk+vi$ if $s < 0$.

(2) $v = 1$ if $a+b+c+2r-2s-2t+2vi-2vj-2vk = 0$.

(d) $v = 1$ if $b+r-vj-vk = 0$ and $s+t = 0$.

(e) $vk > vi$ if $a+b-2r-2s+2t = 0$ and $a-3s = 0$.

(f) $s \geq 2vk$ if $a+b+2r-2s-2t = 0$, $a-3s = 0$ and $a-2vi = 0$.

(g) $v = 1$ if $b+c-2r-2vj+2vk = 0$.

(29). (I) $vi > vj$ if $a = b$ and $a+b+2r+2s+2t = 0$.

(a) $vi+vj+s \geq 0$ if $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$. And $vj > |vi+r| \geq 0$ if $vi+vj+s = 0$.

(b) $v = 1$ if $a+t-vi-vj = 0$.

(II) $v_i \geq v_j$ if $a = b$ and $a+b-2r-2s+2t = 0$.

(a) $v_k \geq v_j$ if $i = j$.

(b) $v_j+v_i \geq r$ if $c-2v_k = 0$. And $s = r$ if $v_i+v_j = r$.

(c) $v = 1$ if $a+t-v_i-v_j = 0$.

(III) $v_i \gtrless v_j$ if $a = b$ and $a+b-2r+2s-2t = 0$.

(a) $v_i+v_j \geq r+t$ if $a+c+2s-2v_i-2v_k = 0$.

(b) $v_i > -s+v_j$ if $a+b+c-2r+2s-2t-2v_i+2v_j-2v_k = 0$.

(IV) $v_i \gtrless v_j$ if $a = b$ and $a+b+2r-2s-2t = 0$.

(a) $v_i \geq -r+v_j$ if $c-2v_k = 0$. And $v_j+v_i \geq s$ if $v_i = -r+v_j$.

(b) $v_i+v_j \geq s+t$ if $b+c+2r-2v_j-2v_k = 0$.

(30). $a+r+s+t > 0$ if $a = b = c$.

(a) $v_i \gtrless v_k \gtrless v_j$ and $|t|>|s|$ if $a+t-r-s = 0$.

(b) $v_j \gtrless v_k$ if $a+s-r-t = 0$.

(1) $v = 1$ if $a+s-v_i-v_k = 0$.

(c) $v_i \gtrless v_k$ if $a+r-s-t = 0$.

(1) $v = 1$ if $a+r-v_j-v_k = 0$.

(31). $v_i \geq v_j$ if $a = b$ and $|r|=|s|$.

(a) $s \geq 0$ if $i = j$ and $r+s = 0$. And $v = 1$ if $r = s$.

(b) $v_i+v_j \geq s \geq 0$ if $c-2v_k = 0$. And if $r < 0$, then $v_i \geq s+v_j$.

(c) $v = 1$ if $a+t-v_i-v_j = 0$ and $r+s = 0$. And $2v_k \geq r+s \geq 0$ if $r = s$.

(d) $v_i \geq |r+t-v_j|$ if $a+c+2s-2v_i-2v_k = 0$. And $a+s-v_i-v_j \leq 0$ if $v_i = |r+t-v_j|$. And $i = j$ if $v_i = |r+t-v_j|$ and $a+s-v_i-v_j = 0$.

(e) $a+r-v_i-v_j \leq 0$ if $b+c+2r-2v_j-2v_k = 0$. And $v_j \geq |t+s-v_i|$ if $a+r-v_i-v_j = 0$.

(f) $a+t+r-v_i-v_j \leq 0$ if $a+b+c+2r+2s+2t-2v_i-2v_j-2v_k = 0$.

(g) $a-r-t-v_i+v_j \leq 0$ if $a+b+c-2r+2s-2t-2v_i+2v_j-2v_k = 0$.

(h) $4v_i \geq a+2v_j$ if $a = b = 2|r| = 2|s| = 2|t|$ and $a+c+2s-2v_i-2v_k = 0$.

(32). $v_i \geq v_j \geq v_k$ if $a = b = c$ and $|r|=|s|=|t|$.

(a) $2v_j \geq r+t \geq 0$ if $a+s-v_i-v_k = 0$. And $v = 1$ if $r+t = 0$.

(b) $a+r+s-v_i-v_k \leq 0$ if $a+b+c+2r+2s+2t-2v_i-2v_j-2v_k$

$=0$.

(c) $a-r-s-vi+vk \leq 0$ if $a+b+c-2r-2s+2t-2vi-2vj+2vk=0$.

(33). $|s| \geq |i|$ if $c=d$.

(a) $|r| \geq |j|$ if $|s|=|i|$.

(b) $t \leq 0$ if $|s|=|i|$ and $|r|=|j|$ and $rs<0$.

(c) $v=1$ if $|s|=|i|$, $|r|=|j|$ and $rs \geq 0$.

(d) $|r|+vj \geq |s|$ if $a-2t=0$ and $|s|=|i|$.

(1) $|s| \geqq 2vj$ if $a+c+2s-2vi-2vk=0$, $a+3s=0$ and $|s|=|i|$.

(e) $|r| \geq |i+j|$ if $a+2t=0$ and $|s|=|i|$.

(34). $a+b+c+2r+2s-2t-2vi+2vj-2vk>0$, $|s|=|j|$ if $a=b$ and $c=d$. And $|r| \geq |i|$ if $|s|=|j|$.

(a) $t \leq 0$ if $|s|=|j|$ and $|r|=|i|$ and $rs<0$. And $s>0$ if $rs<0$ and $t=0$.

(b) $|s| \geq |a+r-vj|$ if $b+c+2r-2vj-2vk=0$. And $|r| \geq |s+t-vi|$ if $|s|=|a+r-vj|$. And $vi=vj$ if $|s|=|a+r-vj|$ and $|r|=|s+t-vi|$.

(c) $|s| \geq |t+r-vj|$ if $a+c+2s-2vi-2vk=0$.

(1) $|r| \geq |a+s-vi|$ if $|t+r-vj|=|s|$.

(2) $vi \geq vj$ if $|t+r-vj|=|s|$ and $|r|=|a+s-vi|$. And $|s|=|r|=|i|=|j|$ if $vi=vj$.

(d) $a+r+t+s-vj \leq 0$ if $a+b+c+2r+2s+2t-2vi-2vj-2vk=0$.

(1) $vi \geq vj$ if $a+r+t+s-vj=0$.

(e) $s \geq -r+vj$ if $c-2vk=0$ and $r<0$. And $vi \geq vj$ if $s=-r+vj$.

(f) $vi+|r| \geq |s|$ if $a-2t=0$ and $|s|=|j|$. And $|i|=|r|$ if $vi+|r|=|s|$.

(g) $-r+vj \geq t+vi$ if $a-2t=0$ and $b+c+2r-2vj-2vk=0$. And $vi-r \geq t$ if $-r+vj=t+vi$.

(1) $-r+t>-2s+vi$ if $a+r+s-vj=0$.

(h) $-2s+vi \geq t-r+vj$ if $a-2t=0$ and $a+c+2s-2vi-2vk=0$. And $a+s+r-vi \leq 0$ if $-2s+vi=t-r+vj$.

(i) $|s| \geq vi+vj$ if $a+2t=0$. And $|r| \geq vi$ and $|r| \geq vj$ if $s=vi+vj$.

(1) $a+r+2s-vi-vj<0$ and $a+r+t+s-vj<0$ if $a+b+c+2r+2s+2t-2vi-2vj-2vk=0$.

(35). $v=1$ if $b=c=d$, $s=t=vi$, and $b+r-vj-vk=0$.

(36). $|t| \geq |k|$ if $a=b=c=d$. And $|r| \geq |i|$ if $|t|=|k|$.

(a) $s < 0$ if $|t|=|k|$, $|r|=|i|$ and $rt < 0$.

(b) $r \leq 0$ if $|t|=|k|$, $|s|=|j|$ and $ts < 0$.

(c) $|s| \geq |s+r-vk|$, $|s| \overline{>} |k|$ if $a+t-vi-vj = 0$.

(1) $v = 1$ if $r = vk$ or if $s = vk$.

(2) $j = k$ if $|s|=|t|$ and $r = vk$.

(3) $v = 1$ if $a+s-vi-vk = 0$.

(d) $a+t-vi-vj-vk \leq 0$ and $a+t+s+r-vk \leq 0$ if $3a+2r+2s+2t-2vi-2vj-2vk = 0$.

(1) $a+4t = 0$ if $|t|=|k|$.

(2) $|s| \geq |k| \geq |r|$ if $a+t-vi-vj-vk = 0$.

(3) $vi \geq vk$ if $a+t+r+s-vk = 0$.

(e) $|i|=|r|=|t|$ if $|t|=|j|=|k|=|s|$ and $a+r-j-k = 0$.

(f) $v = 1$ if $a+s-vi-vk = 0$ and $a+r-vj-vk = 0$.

Transformations are indicated by parentheses. The values of x,y,z and w are given in order in terms of x_1,y_1,z_1 and w_1 but subscripts are omitted since they occur in every case. Following each transformation are the values of the new coefficients R, S, T, I, J, K in terms of the coefficients of ϕ.

PROOF THAT CONDITIONS (8)-(36) CAN BE SATISFIED

To satisfy condition 8, consider the transformations

tr (1) $(x,-y,-z,w) : r,-s,-t,i,-j,-k.$

tr (2) $(-x,y,-z,w) : -r,s,-t,-i,j,-k.$

tr (3) $(-x,-y,z,w) : -r,-s,t,-i,-j,k.$

Case 1. If i,j and k do not have the same sign, use tr (1), (2) or (3) according as i, j or k, respectively, has a sign different from that of the other two, considering terms which are zero as positive.

Now assume that $i \geq 0$, $j \geq 0$, $k \geq 0$ or $i \leq 0$, $j \leq 0$, $k \leq 0$.

Case 2. If $i = 0$ and the first of t,s,j and k not zero is negative, use tr (1).

Now assume that 8(a) is satisfied.

Case 3. If $j = 0$ and the first of t,r,i and k not zero is negative, use tr (2).

Now assume that 8(b) is satisfied.

Case 4. If $k = 0$ and the first of s,r,i and j not zero is negative, use tr (3).

Now assume that 8(c) is satisfied.

Case 5. If $i = j = 0$ and $s \leq 0$ and if $s = 0$ then $r < 0$, use

tr (3) and 8(d) is satisfied.

Let U(x,y,z,1) be a lattice point in the hyperplane parallel to OPQM whose distance from O = ON. By the proofs of theorems 12 and 15, the only possible values of x,y,z are 0,±1. Omitting points excluded by condition 8, all such possible points are :

(1)	(0,0,0,1)	(8)	(-v,-v,-v,1)
(2)	(-v,0,0,1)	(9)	(v,-v,-v,1)
(3)	(0,-v,0,1)	(10)	(-v,v,-v,1)
(4)	(0,0,-v,1)	(11)	(-v,-v,v,1)
(5)	(0,-v,-v,1)	(12)	(0,-v,v,1)
(6)	(-v,0,-v,1)	(13)	(-v,0,v,1)
(7)	(-v,-v,0,1)	(14)	(0,v,-v,1)

Lattice points in the hyperplane OPQM whose distances from O may be equal to ON in case ON is equal to one or more of the distances OM, OQ, OP are :

(15)	(1,1,1,0)	(22)	(0,1,1,0)
(16)	(-1,1,1,0)	(23)	(1,1,0,0)
(17)	(1,-1,1,0)	(24)	(-1,1,0,0)
(18)	(-1,-1,1,0)	(25)	(0,0,1,0)
(19)	(1,0,1,0)	(26)	(0,1,0,0)
(20)	(-1,0,1,0)	(27)	(1,0,0,0)
(21)	(0,-1,1,0)		

The three points (-v,v,0,1), (v,-v,0,1) and (v,0,-v,1) are excluded by conditions 9, 10 and 13.

Conditions 9-36 conform to the following principles :

(1) If a form ϕ which fails to satisfy one of these conditions be replaced by an equivalent form Φ which does; the first, in order, of T, S, R, I, J, K which is different in absolute value from the corresponding coefficient of ϕ is greater than that coefficient.

(2) (a) If the absolute value of each of the corresponding coefficients is the same but RST = -rst $\neq$ 0, then RST > 0.
(b) If the absolute value of each of the corresponding coefficients is the same but RST = rst, then the first in order of the coefficients, as listed in principle 1, to be changed in sign will be made positive.

In view of these two principles, it is sufficient to show that each condition may be satisfied independently of all the others. For suppose in making a transformation to get a form

which satisfies one condition, we obtain a form which fails to satisfy a previous condition. Then we may make a transformation to satisfy that condition and so on. Now, in each such transformation, the values of a,b,c,d remain unchanged and each of the fundamental hyper-parallelopipeds belonging to the equivalent forms is reduced. Therefore conditions 1-4 which determine maximum absolute values for each of the coefficients r,s,t,i,j,k are satisfied. In accordance with principle 1, in each of the transformations mentioned, the first in order of t,s,r,i,j,k to be changed in absolute value will be increased by an integer. Since the absolute values of the coefficients are limited by conditions 1-4, after a finite number of such transformations a form will be obtained for which none of the coefficients t,s,r,i,j,k can be increased in absolute value without decreasing the absolute value of the one which precedes it. Suppose this form violated one of the conditions 9-36 in so far as the absolute value of the coefficients are concerned. Then the transformation made to satisfy this condition would increase the absolute value of one coefficient without changing the absolute value of the preceding coefficients. But this is impossible by hypothesis. Therefore, the condition must be satisfied.

By principle 2, we must, after a finite number of transformations, each of which leaves the absolute value of the coefficients unchanged, obtain a form which also satisfies all conditions concerning the signs of the coefficients.

An examination of the transformations used by Dickson in establishing conditions 5-7a will show that principles 1 and 2a also apply to these conditions, and further, that each of these transformations actually increases the absolute value of one of the coefficients t, s and r. Thus conditions 5-7a can be established without regard to the values of i, j and k, consistent with principle 1.

All conditions are consistent. It has already been shown that the conditions are consistent in so far as the absolute values of the coefficients are concerned. It will be observed that in the transformations used to establish conditions 8-32, the sign of rst is unaffected. In each of these transformations, if t is changed from positive to negative, or if t is unchanged and s is changed from positive to negative, or if t and s remain unchanged and r is changed from positive to negative, then i, j or k

is increased in absolute value in accordance with principle 1. In all transformations in which i, j and k remain unchanged in absolute value, then if t is changed in sign it is from negative to positive, if t is unchanged but s is changed in sign then it is from negative to positive, and if t and s are unchanged and r is changed in sign, then it is from negative to positive. In the transformations used to establish conditions 33-36, the absolute values of the coefficients are unchanged by a transformation, if rst is changed in sign then it is from negative to positive. If rst is also unchanged in sign in one of the latter transformations then the statements concerning the former transformations apply. In every transformation made to satisfy a condition in which v is required to be 1, the coefficients are unchanged with the exception of the signs of i, j and k. In every case where v is required to be 1, the form is improperly equivalent to itself and thus may be called an ambiguous form. Each of the foregoing statements can easily be verified by examination of the transformations used by Dickson, of the transformations used in establishing condition 8, and of the remaining transformations to follow.

Conditions 9-36 are considered in order. Under each condition it is shown how every form which satisfies the previous conditions but in some way violates the condition under consideration may be made to satisfy that condition by a transformation. In many cases, the given condition no longer applies to the form obtained after the transformation. Whenever this is true, the condition which does apply is shown to be satisfied.

Condition 9. If $a-2vi=0$ and if (I) (a) $t \leq 0$; if $t=0$, then $s \leqq 0$: (b) $vk \geq s$; if $vk=s$, then $s=0$ and $r \geqq 0$: use

tr (4) $(x-vw,-y,-z,w) : r,-s,-t,-i,vt-j,vs-k.$

And if (II) (a) $t \leq 0$ and if $t=0$, then $s \geq 2vk$ and if $s=2vk$, then $r<0$: (b) $s \geq vk$ and if $s=vk=0$, then $r=0$: use

tr (5) $(-x-vw,y,-z,w) : -r,s,-t,i,-vt+j,vs-k.$

In each of these cases, note that if $t<0$, then $VJ>T>0$ and by (b), $VK \geq 0$. If $K=0$, then $S \geq 0$. If $S=K=0$, then $R \geq 0$. If $T=0$, then by (b), $2VK \geq S \geqq 0$. And if $T=0$, $2VK=S$, then $R \geq 0$. Similarly, if $a-2vi=0$ and if (III) (a) $t \geq 2vj$ and if $t=2vj$, then $s \geq 2vk$: (b) $s \geq vk$ and if $s=k=0$, then $r>0$: use

tr (6) $(x-vw,y,z,w) : r,s,t,-i,-vt+j,-vs+k.$

And if $a-2vi=0$ and if (IV) (a) $t \geq 2vj$ and if $t=2vj$, then $s \leq 0$ and if $s=0$, then $r<0$: (b) $k \geq s$ and if $vk=s$, then

$s = k = 0$ and $r \leq 0$; use

tr (7) $(-x-vw, -y, z, w) : -r, -s, t, i, vt-j, -vs+k.$

In each of these last two cases, if $t > 0$, then $2VJ \geq T$. By (b), $VK \geq 0$, and if $K = 0$, then $S > 0$ or $S = 0$, $R > 0$. By (b), if $T = 2VJ$, then $2VK \geqq S \geq 0$, and if $S = 0$, then $R \geq 0$. Thus condition 9 is established.

Condition 10. With the set of transformations listed below, it may be shown that condition 10 can be satisfied. Since the hypotheses under which these transformations are used are exactly analogous to those used to establish condition 9, they are not listed.

tr (8) $(-x, y-vw, -z, w) : -r, s, -t, vt-i, -j, vr-k.$

tr (9) $(x, -y-vw, -z, w) : r, -s, -t, -vt+i, j, vr-k.$

tr (10) $(-x, -y-vw, z, w) : -r, -s, t, vt-i, j, -vr+k.$

tr (11) $(x, y-vw, z, w) : r, s, t, -vt+i, -j, -vr+k.$

Condition 11. If $a+b+2t-2vi-2vj = 0$ and if (I) (a) $s+r \geq vk$; if $s+r = vk$, then $s = 0$; and if $s = 0$, then $r > 0$: (b) $a+t-2vi \geq 0$; and if $a+t-2vi = 0$, then $s+r \geq 2vk$: use

tr (12) $(x-vw, y-vw, z, w) : r, s, t, -va-vt+i, -vb-vt+j, -vs-vr+k.$

Since $a+t-2vi \geq 0$, then $A+T-2VI \leq 0$, and if $K = 0$, by (a), $S \geq 0$. And if $S = 0$, $R \geq 0$. If $S+R = 2VK$ and $A+T-2VI = 0$, then $V = 1$.

If $a+b+2t-2vi-2vj = 0$ and if (II) (a) $vk \geq s+r$; if $vk = s+r$, then $s \leq 0$; and if $s = 0$, then $s = r = k = 0$: (b) $a+t-2vi \geq 0$; if $a+t-2vi = 0$, then $s+r \leq 0$; if $s+r = 0$, then $s \leq 0$: use

tr (13) $(-x-vw, -y-vw, z, w) : -r, -s, t, va+vt-i, vb+vt-j, -vr-vs+k.$

As before, $A+T-2VI \leq 0$, and if $K = 0$, then $S \geq 0$. If $A+T-2VI = 0$, then $2VK \geq S+R \geq 0$. If $S+R = 0$, then $S \geq 0$. Hence condition 11 is satisfied.

Condition 12. If $c-2vk = 0$, then use transformations (14), (15), (16) or (17) under hypotheses exactly analogous to those for transformations (4), (5), (6) and (7).

tr (14) $(-x, -y, z-vw, w) : -r, -s, t, vs-i, vr-j, -k.$

tr (15) $(x, -y, -z-vw, w) : r, -s, -t, -vs+i, vr-j, k.$

tr (16) $(-x, y, -z-vw, w) : -r, s, -t, vs-i, -vr+j, k.$

tr (17) $(x, y, z-vw, w) : r, s, t, -vs+i, -vr+j, -k.$

Condition 13. If $a+c+2s-2vi-2vk = 0$, then use transformation (18) or (19) under hypotheses exactly analogous to those for transformations (12) and (13). Note that $c+s-vk > 0$, since condition 12 is now satisfied.

tr (18) $(x-vw, y, z-vw, w) : r, s, t, -va-vs+i, -vt-vr+j, -vc-vs+k.$

tr (19) $(x-vw, y, z-vw, w) : -r, s, -t, va+vs-i, -vt-vr+j, vc+vs-k.$

Condition 14. If $b+c+2r-2vj-2vk = 0$ and if (I) $s+t \geq 2vi$; if $s+t = 2vi$, then $b+r-2vj \geq 0$: use

tr (20) $(x, y-vw, z-vw, w) : r, s, t, -vt-vs+i, -vb-vr+j, -vc-vr+k.$

If $b+r-vj = 0$, then $b+2r = 0$, $b-2vj = 0$ and, by condition (10), $t > 0$. Hence, if $J = 0$, condition (8) is satisfied. If $c+r-vk = 0$, then $c-2vk = 0$ and, by condition (12), $s > 0$. Hence if $K = 0$, condition (8) is satisfied. Since $t+s-2vi \geq 0$, must $2VI \geq S+T \geq 0$ and if $S+T-2VI = 0$, then $B+R-2VJ \neq 0$. Hence, condition (14) is satisfied.

And if $b+c+2r-2vj-2vk = 0$ and if (II) (a) $s+t \leq 0$; if $s+t = 0$ then $b+r-2vj \geq 0$ and either $vi > 0$ or $i = s = t = 0$: (b) if $b+r-2vj = s+t = 0$, then $t < 0$: (c) $b+r-vj > 0$, $c+r-vk > 0$: use

tr (21) $(x, -y-vw, -z-vw, w) : r, -s, -t, -vt-vs+i, vb+vr-j, vc+vr-k.$

By (a) $2VI \geq S+T \geq 0$ and $B+R-2VJ \leq 0$ if $S+T = 0$. By (a), $S = T = 0$ if $I = 0$. By (b), $T > 0$ if $S+T = B+R-2VJ = 0$. By (c), $VK > 0$.

Also if $b+c+2r-2vj-2vk = 0$ and if (III) $b+r-vj = 0$, $s+t < 0$, use

tr (22) $(-x, -y-vw, z-vw, w) : -r, -s, t, vt+vs-i, 0, -vc-vr+k.$

Evidently $B+C-2R+2VJ-2VK = C-2VK = B-2R = J = 0$. Now $2VI-S = -2s+2vi-2t+s = 2vi-s-2t = (2vi-t)-(s+t)$, of which each parenthesis is positive so that $2VI > S > T > 0$, and conditions 8b and 19 are satisfied.

Condition 15. If $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$ and if (I) (a) $a+t+s-2vi \geq 0$; if $a+t+s-2vi = 0$, then $b+r+2t-2vj \geq 0$: (b) $b+r+t-vj > 0$: (c) $c+r+s-vk > 0$: use

tr (23) $(x-vw, y-vw, z-vw, w) : r, s, t, -va-vt-vs+i, -vb-vt-vr+j, -vc-vr-vs+k.$

We have $A+B+C+2R+2S+2T-2VI-2VJ-2VK = 0$. By (a), $A+T+S-2VJ \leq 0$. If $A+T+S-2VI = 0$, then $B+R+T-2VJ \neq 0$. By (a), (b) and (c), conditions 8 and 15 are satisfied.

And if $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$ and if (II) (a) $a+t+s-2vi > 0$: (b) $b+r+t-vj \leq 0$: use

tr (24) $(-x-vw, y-vw, z-vw, w) : -r, s, -t, va+vt+vs-i, -vb-vt-vr+j, vc+vr+vs-k.$

Since $b+r+t-vj \leq 0$, then $c+r+s-vk > 0$. Were $b+r+t-vj \leq 0$ and $c+r+s-vk \leq 0$, then since $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$, it would follow that $b+c+2r-2vj-2vk < 0$, contrary to condition 3. Therefore, $c+r+s-vk > 0$ and $VK > 0$. Thus $A+B+C-2R+2S-2T+2VJ-2VI-2VK = 0$ and $A+S-T-2VI < 0$, by (a). If $J = 0$, then $T > 0$, so condi-

tions 8 and 17 are satisfied.

And if $a+b+c+2r+2s+2t-2vi-2vj-2vk=0$ and if (III) (a) $a+t+s-2vi \geq 0$: (b) $c+r+s-vk \leq 0$: use

tr (25) $(-x-vw,-y-vw,+z-vw,w)$: $-r,-s,t,va+vt+vs-i,vb+vt+vr-j,-vc-vr-vs+k$.

Therefore $A+B+C-2R-2S+2T-2VI-2VJ+2VK=0$ and for reasons analogous to those used in the last case, $A-S+T-2VI \leq 0$. By (b), if $K=0$, then $S>0$. Thus conditions 8 and 18 are satisfied.

Condition 16. If $a+b+c+2r-2s-2t+2vi-2vj-2vk=0$ and if (I) (a) $a-s-t>0$: (b) $b+r-t-vj>0$: (c) $c+r-s-vk>0$: use

tr (26) $(x+vw,-y-vw,-z-vw,w)$: $r,-s,-t,va-vt-vs+i,vb-vt+vr-j,vc-vs+vr-k$.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK=0$. By (a), $A+T+S-2VI=a-s-t-2a+2t+2s-2vi=-(a-s-t)-2vi<0$. Therefore, condition 15 is satisfied. By (b) and (c), condition 8 is satisfied.

And if $a+b+c+2r-2s-2t+2vi-2vj-2vk=0$ and if (II) (a) $a-s-t>0$: (b) $b+r-t-vj \leq 0$: use

tr (27) $(-x-vw,-y-vw,z-vw,w)$: $-r,-s,t,-va+vt+vs+i,vb-vt+vr-j,-vc-vr+vs+k$.

Thus $A+B+C-2R+2S-2T-2VI+2VJ-2VK=0$ and $A+T+S-2VI<0$. Condition 8 is satisfied.

And if $a+b+c+2r-2s-2t+2vi-2vj-2vk=0$ and if (III) (a) $a-s-t>0$: (b) $c+r-s-vk \leq 0$: use

tr (28) $(-x+vw,y-vw,-z-vw,w)$: $-r,s,-t,-va+vt+vs-i,-vb+vt-vr+j,vc+vr-vs-k$.

Thus $A+B+C-2R-2S+2T-2VI-2VJ-2VK=0$ and $A+T-S-2VI<0$ and condition 8 is satisfied.

And if $a+b+c+2r-2s-2t+2vi-2vj-2vk=0$ and if (IV) (a) $a-s-t=0$; (b) $b+r-2vj-s+vi>0$: use

tr (29) $(x+y+z+vw,-y-vw,-z-vw,w)$: $r,s,t,i,vb+vr-vs-j+i,vc+vr-vs-k+i$.

Since $r>0$ and $vi \geq 0$, condition 8 is satisfied. Since $B+R-S-2VJ+VI=-b-r+s+2vj-vi<0$, condition 16 is satisfied.

Condition 17. If $a+b+c-2t+2s-2r-2vi+2vj-2vk=0$ and if (I) (a) $a-t+s-2vi \geq 0$: (b) $c-t-r-vk>0$: use

tr (30) $(-x-vw,y+vw,-z-vw,w)$: $-r,s,-t,va-vt+vs-i,vb-vt-vr+j,vc-vr+vs-k$.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK=0$ and $A+T+S-2VI \leq 0$ by (a). If $A+T+S-2VI=0$, then $B+T+R-2VK=-(b-t-r+2vj)<0$. Were the latter expression zero, then $b-t-r=0$ and therefore, $s>0$ and

$a+c-2r-2vi-2vk = a-2vi = k = 0$, contrary to condition 10.

And if $a+b+c-2t+2s-2r-2vi+2vj-2vk = 0$ and if (II) (a) $a-t+s-2vi \geq 0$: (b) $c+s-r-vk \leq 0$: use

tr (31) $(x-vw,-y+vw,-z-vw,w)$: $r,-s,-t,-va+vt-vs+i,-vb+vt+vr-j,$ $vc-vr+vs-k$.

Thus $A+B+C-2R-2S+2T-2VI-2VJ+2VK = 0$. Since $c+s-r-vk \leq 0$, then $s < 0$ and $\sigma = -1$. Were $s = 0$, then $c-2r = 0$, and since $b-2t-2r+2vj \leq 0$, $t > 0$, contrary to condition 6. Conditions 8 and 18 are therefore satisfied.

Condition 18. If $a+b+c+2t-2s-2r-2vi-2vj+2vk = 0$ and if (I) (a) $a+t-s-2vi > 0$: (b) $b+t-r-vj > 0$: use

tr (32) $(-x+vw,-y-vw,z+vw,w)$: $-r,-s,t,va+vt-vs-i,vb+vt-vr-j,$ $vc-vr-vs-k$.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK = 0$ and $A+T-S-2VI < 0$. By (b), $VJ > 0$. Were $K = 0$, then $c-r-s = 0$ and $a = b = c = 2r = 2s = 2t$ and $a+b+c-2r-2s-2j = a+t-s-2vi = 0$, contrary to (a).

And if $a+b+c+2t-2s-2r-2vi-2vj+2vk = 0$ and if (II) (a) $a+t-s-2vi > 0$: (b) $b+t-r-vj \leq 0$: use

tr (33) $(x-vw,-y-vw,-z+vw,w)$: $r,-s,-t,-va-vt+vs+i,vb+vt-vr-j,$ $-vc+vr+vs-k$.

Thus $A+B+C-2R+2S-2T-2VI+2VJ-2VK = 0$, and by (a), $A+T-S-2VI < 0$. By (b), $VJ \geq 0$ and $T > 0$. By the same reasoning as in the previous case, $VK > 0$.

Condition 19. If $b+c-2r+2vj-2vk = 0$, and therefore $j = 0$, and if (I) (a) $t \geq 2vi > s > 0$: (b) $c > b$: use

tr (34) $(-x,-y+vw,z-vw,w)$: $-r,-s,t,-vt+vs-i,-vb-vr-j,$ $-vc+vr+k$.

Thus $B-2VJ = 0$, $B+C+2R-2VJ-2VK = 0$ and $2VI-T = (2vi-s)+(t-s)$, of which both parentheses are positive by (a) and $VK > 0$ by (b). Since $2VI > S+T > 0$, conditions 8 and 14 are satisfied.

And if $b+c-2r+2vj-2vk = 0$ and if (II) (a) $t \geq s > 0$: (b) $b = c$: use the transformation

tr (35) $(x,y+vw,z-vw,w)$: $r,s,t,vt-vs+i,vb-vr+k,0$.

Thus $B+C-2R-2VJ+2VK = 0$ and, therefore, $B = C = 2R = 2VJ$, $K = 0$. Since $c-2vk = 0$ and $j = 0$, then $2vi > s > 0$. Now $2VI-T = (2vi-s)+(t-s)$ of which the first parenthesis is positive, the second positive or zero, so that $2VI > T > 0$.

Condition 20. If $a-t-s = 0$ and if (I) (a) $vi \geq 2vj$; if $i = 2j$, then $vi > 2vk$: (b) $vi \geq vk$: (c) if $i = 0$, then $v = -1$: use

tr (36) $(x+y+z,-y,-z,w)$: $r,s,t,i,i-j,i-k$.

By (a) and (b), $VI \geq 0$, $VJ \geq 0$, $VK \geq 0$ and R, S and $T > 0$. By (a) and (b), $2VJ \geq VI$ and if $2VJ = I$, then $2VK \geq VI$ so that conditions 8, 20, and 20c are satisfied.

And if $a-t-s = 0$ and if (II) (a) $vk > vi > 2vj$: use

tr (37) $(-x-y+z, y, -z, w)$: $-r, -s, t, -i, -i+j, -i-k$.

By (a), $2VJ > VI$ and $VK > 0$, so that conditions 8 and 20 are satisfied.

If $a+t+s = 0$: use

tr (38) $(x+y+z, y, z, w)$: $r, -s, -t, i, i+j, i+k$.

Conditions 8 and 20a are obviously satisfied.

If $a+t-s = 0$ and if (I) (a) $vk \leq vi$: use

tr (39) $(-x-y-z, -y, z, w)$: $-r, s, -t, -i, -i-j, -i+k$.

By (a), condition 8 is satisfied. Obviously condition 20 is satisfied.

And if $a+t-s = 0$ and if (II) $vk > vi$: use

tr (40) $(x+y-z, y, z, w)$: $r, -s, -t, i+j, -i+k$.

As before, conditions 8 and 20 are satisfied.

If $a-t+s = 0$ and if (I) $vi \geq 2vj$: use

tr (41) $(-x-y-z, y, -z, w)$: $-r, -s, t, -i, -i+j, -i-k$.

As in previous cases, conditions 8 and 20 are satisfied.

And if $a-t+s = 0$ and if (II) $i = 0$, $v = -1$: use

tr (42) $(x+y-z, -y, -z, w)$: $r, s, t, i, i-j, -i-k$.

Since $i = 0$, $|j| = |J|$, $|k| = |K|$, and $V = 1$, condition 20c is satisfied.

If $a-t-s = 0$ and $a+2i = 0$: use

tr (43) $(x+y+z+w, -y, -z, w)$: $r, s, t, -i, -j, -k$.

Conditions 8 and 20c are satisfied.

If $a-t+s = 0$ and if (a) $a+c+2s-2vi-2vk = 0$: (b) $a+s-2vi = 0$: use

tr (44) $(-x-y-z-vw, y, -z-vw, w)$: $-r, -s, t, i, -vr-i+j, vc-i-k$.

Thus $C-2VK = 0$ and $2VI = S > 0$ by (b). Therefore $2VJ-R = -r-2vi+2vj = -r-t+2vj \geq 0$, by condition 13a. Hence condition 12 is satisfied. Since $2|r| \geq |s| = 2vi$, then $|J| \geq |j|$ and $|K| \geq |k|$.

If $a-t+s = 0$ and $b+c+2r-2vj-2vk = 0$ and (a) $b+r+vi-2vj > 0$: (b) $c+r-i-k > 0$: use

tr (45) $(x+y-z, -y-vw, -z-vw, w)$: $r, s, t, i, vb+vr+i-j, vc+vr-i-k$.

By (a), $B+R+VI-2VJ \leq 0$ and by (b), $VK > 0$ so that condition 20 is satisfied.

If $a-t+s = 0$ and $b+c+2r-2vj-2vk = 0$ and (a) $c+r-i-k = 0$: use

tr (46) $(-x-y-z, y-vw, -z-vw, w)$: $-r, -s, t, -i, -vb-vr-i+j, vc+vr-i-k$.

Thus $B+C-2R-2VJ+2VK=0$. Therefore $K=0$, $B=C=2R=2J$. Since $a+c+2s-2vi-2vk \geq 0$ and $a+2s=0$, then $c-2vi-2vk \geq 0$. But since $c+2r=0=c+r-vi-vk$, then $c-2vi-2vk=a+c+2s-2vi-2vk=0$. By condition 20b, $a+s-2vi<0$, hence $2VI>T>0$ and condition 10 is satisfied.

Condition 21. If $a-2t=0$, $|r|=|s-r|$ and (a) $vi \geq 2vj$; if $i=2j$ and $2r=s$, then $v=-1$; if $i=2j$ and $s=0$, then $r \leqq 0$: use

tr (47) $(-x-y,y,-z,w)$: $s-r,s,t,-i,-i+j,-k$.

Thus $2VJ \geq VI$ and if $2VJ=VI$ and $2R=S$, then $V=1$. If $s=0$, then $R \overline{\geq} 0$, so that condition 21 is satisfied.

If $a-2t=0$ and $|r|=|s-r|$ and (a) $i=0$: (b) $s \leqq 0$: use

tr (48) $(-x-y,y,z,-w)$: $r-s,-s,t,0,-j,-k$.

Thus $S \overline{\geq} 0$ and conditions 21-Ib and 21-IIb are satisfied.

If $a+2t=0$ and $|r|=|s-r|$: use

tr (49) $(x+y,y,z,w)$: $r+s,s,-t,i,i+j,k$.

Condition 21a is obviously satisfied.

If $a-2t=0$, $a-2vi=0$ and (a) $2r=s<0$ or (b) $s=0$, $v=-1$: use

tr (50) $(x+y-vw,-y,-z,w)$: $r-s,-s,t,-i,-j,-vs-k$.

Condition 21b is satisfied.

If $a-2t=0$, $a-2vi=0$ and $2r=s>2vk$: use

tr (51) $(-x-y-vw,y,-z,w)$: $r,s,t,i,j,vs-k$.

Condition 21-Ia is satisfied.

If $c-2vk=0$, $a-2t=0$, $s=0$ and $2vj<vi+r$: use

tr (52) $(x+y,-y,-z-vw,w)$: $r,s,t,i,vr+i-j,k$.

Condition 21-IIc is satisfied.

Condition 22. If $a-2s=0$ and $|r|=|t-r|$ and (a) $vi \geq 2vk$; if $i=2k$ and $2r=t$, then $v=-1$; if $t=0$, then $r \leqq 0$: use

tr (53) $(-x-z,-y,z,w)$: $t-r,s,t,-i,-j,-i+k$.

Condition 22a is satisfied.

If $a+2s=0$ and $|r|=|r+t|$: use

tr (54) $(x+z,y,z,w)$: $t+r,-s,t,i,j,i+k$.

Condition 22 is satisfied.

If $a-2s=0$, $t=i=0$ and $v=-1$: use

tr (55) $(x+z,-y,-z,w)$: $r,s,t,i,-j,-k$.

Condition 22c is satisfied.

If $a-2s=0$, $t=0$ and $a+2i=0$: use

tr (56) $(x+z+w,-y,-z,w)$: $r,s,t,-i,-j,-k$.

Then $V=1$.

If $b-2vj=0$, $a-2s=0$, $t=0$ and $2vk<vi+r$: use

tr (57) $(x+z,-y-vw,-z,w) : r,s,t,i,j,vr+i-k.$

Then 22-IIc is satisfied.

Condition 23. If $b-2r=0$, $|s|=|t-s|$ and if (a) $vj \geq 2vk$; if $j=2k$ and $2s=t$, then $v=-1$ and if $t=0$, then $s \leqq 0$: use

tr (58) $(-x,-y-z,z,w) : r,t-s,-i,-j,-j+k.$

Condition 23a is satisfied.

If $b+2r=0$, $|s|=|t+s|$: use

tr (59) $(x,y+z,z,w) : -r,t+s,t,i,j,j+k.$

Condition 23a is satisfied.

If $b+2j=0$, $b-2r=0$ and $t=0$: use

tr (60) $(-x,y+z+w,-z,w) : r,s,t,-i,-j,k.$

Condition 23-IIc is satisfied.

If $a-2vi=0$, $b-2r=0$, $t=0$ and $2vk<vj+s$: use

tr (61) $(-x-vw,y+z,-z,w) : r,s,t,-i,j,-k+vs+j.$

Condition 23-IIb is satisfied.

If $b-2r=0$, $t=j=0$ and $v=-1$: use

tr (62) $(-x,y+z,-z,w) : r,s,t,-i,j,-k.$

Condition 23-IIc is satisfied.

Condition 24. If $a+b+2r+2s+2t=0$ and $a+t+2s=0$: use

tr (63) $(x+z,y+z,z,w) : -r,-s,t,i,j,i+j+k.$

Thus $A+B-2R-2S+2T=0$ and $2VK>VI+VJ$.

Condition 25. If $a+b-2r-2s+2t=0$, $a+t-2s=0$ and (a) $vi+vj \geq 2vk$: use

tr (64) $(-x-z,-y-z,z,w) : r,s,t,-i,-j,-i-j+k.$

Thus $A+B-2R-2S+2T=0$ and condition 25 is satisfied.

Condition 26. If $a+b-2r+2s-2t=0$, $a-t+2s=0$ and (a) $vj \geq 2vk+vi$: use

tr (65) $(-x+z,-y-z,z,w) : r,s,t,-i,-j,i-j+k.$

Thus $A+B-2R+2S-2T=0$ and condition 26 is satisfied.

If $a+b-2r+2s-2t=0$, $a-t+2s=0$ and (a) $vi \geq vj$: use

tr (66) $(x+z,y-z,z,w) : -r,-s,t,i,j,i-j+k.$

Thus $A+B+2R-2S-2T=0$, $A-2S-T=0$ and $2VK+VI>VI \geq VJ$, hence condition 27 is satisfied.

Condition 27. If $a+b+2r-2s-2t=0$, $a-2s-t=0$ and (a) $vi \geq 2vk+vj$: use

tr (67) $(-x-z,-y+z,z,w) : r,s,t,-i,-j,-i+j+k.$

Thus $A+B+2R-2S-2T=0$ and condition 27 is satisfied.

If $a+b+2r-2s-2t=0$, $a-2s-t=0$ and $vj>vi$: use

tr (68) $(x-z,y+z,z,w) : -r,-s,t,i,j,-i+j+k.$

Thus $A+B-2R+2S-2T = 0$ and $2VK+VI > VJ > VI$ and condition 26 is satisfied.

Condition 28. If $b = c$, $|s|=|t|$ and (a) $vk \geq vj$: (b) if $k = j$, either $t = -s < 0$ or $t = s$, $v = -1$: use

tr (69) $(-x,-z,-y,w)$: $r,t,s,-i,-k,-j$.

Condition 28a is satisfied.

If $b = c$, $t+s = 0$, $i = 0$ and $vk \geq vj$: use

tr (70) $(-x,z,y,w)$: $r,-s,-t,-i,k,j$.

Thus $I = 0$, $T \geq 0$ and if $T = 0$, then $S = 0$ so that conditions 8 and 28 are satisfied.

If $t = s = r = j = 0$ and $k > 0$: use

tr (71) $(x,-z,y,w)$: $-r,-t,s,i,k,-j$.

Thus condition 28 is satisfied.

If $a-2vi = 0$, $b = c$, $s = t > vj+vk$: use

tr (72) $(-x-vw,-z,-y,w)$: $r,t,s,i,vs-k,vt-j$.

Condition 28b is satisfied.

If $a-2vi = 0$, $s < 0$, $vj < vk+t$ and $vj > t$: use

tr (73) $(-x-vw,z,y,w)$: $r,s,t,i,-vs+k,-vt+j$.

Condition 28b is satisfied.

If $a-2vi = 0$, $s < 0$, $vj < vk+t$ and $vj \leq t$: use

tr (74) $(x-vw,z,-y,w)$: $-r,-s,t,-i,vs-k,vt+j$.

Condition 28b is satisfied.

If $a-t-s = 0$, $vi \geq vj+vk$: use

tr (75) $(-x-y-z,z,y,w)$: $r,s,t,-i,-i+k,-i+j$.

Condition 28c is satisfied.

If $a-t+s = 0$, $vj \leq vi+vk$ and $vj > vi$: use

tr (76) $(-x-y+z,-z,-y,w)$: $r,s,t,-i,-i-k,i-j$.

Condition 28-c2 is satisfied.

If $a-t+s = 0$ and $vi \geq vj$: use

tr (77) $(x+y+z,-z,y,w)$: $-r,-s,t,i,k+i,i-j$.

Condition 28c is satisfied.

If $a-t-s = 0$ and $a+b+c+2r-2s-2t-2i+2j+2k = 0$: use

tr (78) $(-x-y-z-w,z+w,y+w,w)$: $r,s,t,-i,-j,-k$.

Then $V = 1$.

If $b+r+j+k = 0$ and $t+s = 0$: use

tr (79) $(-x,z+w,y+w,w)$: $r,s,t,-i,-j,-k$.

Thus $V = 1$, $B+R-J-K = 0$, in accordance with condition 28d.

If $a+b+2t-2r-2s = 0$, $a+3t = 0$, $s = -t$ and (a) $vi > vk$: use

tr (80) $(x+y,y+z,-y,w)$: $r,-s,-t,i,i+j-k,j$.

Thus $A+B+2S-2T-2R = 0$, $A-3T = 0$, $S = -T$ and $vj > vk$ since

$t < 0$. Therefore $VJ > VI$ and conditions 26 and 28 are satisfied.

If $a-2vi = 0$, $a+b+2r-2s-2t = 0$, $a-3s = 0$ and $2vk > s$: use

tr (81) $(-x-y-vw, y, y-z, w)$: $r, s, t, i, k+j-va/6, vs-k$.

Thus $2VK+VJ-VI = vj-vk \geq 0$ and $2VK < S$ so that condition 28f is satisfied.

If $b+c-2r+2j-2k = 0$ and $k = 0$: use

tr (82) $(-x, -z+w, -y-w, w)$: $r, s, t, -i, -j, -k$.

Thus $B+C-2R-2J+2K = 0$, $K = 0$ and condition 28g is satisfied.

Condition 29. If $a = b$ and $a+r+s+t = 0$ and $vj \geq vi$: use

tr (83) $(-y-z, -x-z, -z, w)$: $-r, -s, t, -j, -i, -i-j-k$.

Thus $A+T-R-S = 0$ and $VI \geq VJ$ so that condition 29-II is satisfied.

If $a+r+s+t = 0$ and $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$ and if (I) (a) $vi+vj+s \leq 0$; if $vi+vj+s = 0$, then $r = s$: (b) $r+vi < 0$: use

tr (84) $(-y-z-vw, -x-z-vw, -z-vw, w)$: $-r, -s, t, -vs-j, -vr-i, vc/2$.

Thus $C-2VK = 0$, $A-R-S+T = 0$. Now $vi > vj$, by condition 29, and $2VI-S = -s-2vj < 0$, by (a), so condition 12 is satisfied. By (a), $VJ+VI-R = -s-vi-vj \geq 0$, so condition 29-II is satisfied.

And if $a+r+s+t = 0$ and $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$ and if (a) $vj+vi+s \leq 0$; if $vi+vj+s = 0$, then $vi+r-vj \geq 0$: (b) $vi+r > 0$: use

tr (85) $(-y+z-vw, x+z-vw, z-vw, w)$: $r, -s, -t, vs+j, -vr-i, -vc/2$.

Thus $C-2VK = 0$ and $A+R-S-T = 0$ and $VI+R-VJ = vi+vj+s \leq 0$. If $vj+vi+s = 0$, then $VJ+VI-S = vi+r-vj \geq 0$, so condition 29-IV is satisfied.

If $a+t+r+s = 0$, $a+t+i+j = 0$: use

tr (86) $(y-z+w, x-z+w, -z, w)$: $r, s, t, -i, -j, -k$.

Thus $V = 1$ and condition 29-Ib is satisfied.

If $a-r-s+t = 0$ and (a) $vj \geq vi$; if $j = i$, then $vi \geq vk$: (b) $vi+vj \geq k$: use

tr (87) $(y+z, x+z, -z, w)$: $r, s, t, j, i, i+j-k$.

By (a) $VI \geq VJ$ and if $I = J$, then $VK \geq VJ$, so condition 29-II is satisfied.

If $a-r-s+t = 0$ and (a) $vj > vi$: (b) $vk > vj+vi$: use

tr (88) $(-y+z, -x+z, -z, w)$: $-r, -s, t, -j, -i, i+j-k$.

Thus $A+R+S+T = 0$ and $VI > VJ$ so that condition 29-I is satisfied.

If $a-r-s+t = 0$ and $c-2vk = 0$ and if (a) $vi+vj+r \leq 0$; if $vi+vj+r = 0$, then $s > r$: use

tr (89) (-y+z,-x+z,-z-vw,w) : -r,-s,-t,vr-j,vs-i,vc-vr-vs-k.

Thus A+R+S+T = 0, A+B+C+2R+2S+2T-2VI-2VJ-2VK = 0 and VI>VJ. For r-vj ≥ vi > s-i. Were 2vi = s, then 2vj > r so that s = r = 2vi = 2vj, contrary to (b). By (a), S+VJ+VI = r-vi-vj ≤ 0. If r+vi +vj = 0, then -R-VI = vj and VJ+R+VI = s-vi-vj > 0, so that condition 29-Ia is satisfied.

If a-r-s+t = 0, and a+t+i+j = 0 : use

tr (90) (y+z+w,x+z+w,-z,w) : r,s,t,-i,-j,-k.

So V = 1 and condition 29-IIc is satisfied.

If a-r+s-t = 0 and if (I) (a) vj ≧ vi : (b) vi+vk > vj : use

tr (91) (-y-z,-x+z,-z,w) : r,s,t,-j,-i,-i+j-k.

Thus A-R+S-T = 0 and VI ≠ VJ so that condition 29-III is satisfied.

And if a-r-s+t = 0 and if (II) (a) vj ≥ vk+vi : use

tr (92) (y-z,x+z,-z,w) : -r,-s,t,j,-i+j-k.

Thus A+R-S-T = 0, VI > VJ, so that condition 29-IV is satisfied.

If a-r+s-t = 0 and a+c+2s-2vi-2vk = 0 and t+r > vi+vj : use

tr (93) (-y-z-vw,-x+z,-z-vw,w) : r,s,t,vt+vr-j,va+vs-i, vc-i+j-k.

Thus A-R+S-T = 0, A+C+2S-2VI-2VK = 0 and VI+VJ-T-R > 0, so that condition 29-III is satisfied.

If a+b+c-2r+2s-2t-2vi+2vj-2vk = 0 and if (I) vi < r : use

tr (94) (y+z-vw,-x-z+vw,z-vw,w) : r,-s,-t,vs-j,-vr+i,-vc/2.

Thus A-R-S+T = 0, C-2VK = 0 and VI ≥ S, hence condition 29-IIb is satisfied.

And if a+b+c-2r+2s-2t-2vi+2vj-2vk = 0 and if (II) r ≤ vi ≤ -s+vj : use

tr (95) (y-z-vw,x+z+vw,-z-vw,w) : -r,-s,t,-vs+j,-vr+i,vc/2.

Thus A+R-S-T = 0, C-2VK = 0 and VI+R-VJ = -s+vj-vi ≥ 0. And if vi = vk-s, then VJ+VI-S = vi+vj-r ≥ 0, since vi ≥ -s ≥ r, so that condition 29-IVa is satisfied.

If a+r-s-t = 0 and (a) vj ≥ vi : use

tr (96) (-y+z,-x-z,-z,w) : r,s,t,-j,-i,i-j-k.

Now, A+R-S-T = 0, so that condition 29-IV is satisfied.

If a+r-s-t = 0, c-2vk = 0 and (I) (a) -r+vj ≥ vi; if -r+vj = vi, then s > vi+vj : (b) s > vi : use

tr (97) (y-z,-x+z,z-vw,w) : r,-s,-t,vr-j,-vs+i,-vc-vr+vs+j-i+k.

Thus A+R+S+T = 0, A+B+C+2R+2S+2T-2VI-2VJ-2VK = 0. Now, VI+VJ+S = -r-vi+vj ≤ 0, by (a). If -r-vi+vj = 0, then VI ≥ -R, VI

+R-J = vj+vi-s < 0, hence condition 29-Ia is satisfied.

And if a+r-s-t = 0, c-2vk = 0 and (II) if -r+vj > vi ≥ s : use

tr (98) (y+z,x-z,-z-vw,w) : -r,-s,t,-vr+j,-vs+i,+vc+vr-vs-j+i-k.

Thus A+B+C-2R+2S-2T-2VI+2VJ-2VK = 0 and VI+S-VJ = -r+vj-vi < 0, by (a). Hence condition 29-IIIb is satisfied.

If a+r-s-t = 0, b+c+2r-2vj-2vk = 0 and vi+vj < s+t : use

tr (99) (-y+z,-x-z-vw,-z-vw,w) : r,s,t,vt+vs-j,vt+vs-i, vc+vr-k-j+i.

Thus B+C+2R-2VJ-2VK = 0 and VI+VJ > S+T, so that 29-IVb is satisfied.

Condition 30. If a = b = c and a+r+s+t = 0 : use

tr (100) (-x,-x-z,-x-y,w) : r,-s,-t,-i-j-k,-k,-j.

Thus A+R-S-T = 0 and VI > VJ and VI > VK, so that condition 30 is satisfied.

If a-r-s+t = 0 and (a) vk ≥ vi : (b) vi+vj > vk : use

tr (101) (-y-z,-y,-x+y,w) : r,s,t,-k,-i-j-k,-i.

Now VI ≥ VK and since vi ≷ vj, by condition 29-II, VK > VJ, so that condition 30a is satisfied.

If a-r-s+t = 0 and vk ≥ vi+vj : use

tr (102) (-y+z,-y,x+y,w) : -r,s,-t,k,-i-j+k,i.

Thus A+R-S-T = 0 and VI > VJ, VI > VK so that condition 30c is satisfied.

If a-r-s+t = 0 and vi ≥ vj ≥ vk : use

tr (103) (-x,-x-z,x-y,w) : r,s,t,-i-j+k,-k,-j.

Thus VI ≷ VK ≷ VJ and A-R-S+T = 0, so that condition 30a is satisfied.

If a-r+s-t = 0 and (a) vk ≥ vj : (b) vi > vj : use

tr (104) (-x,x-z,-x-y,w) : r,s,t,-i+j-k,-k,-j.

Thus A-R+S-T = 0 and VI ≷ VJ ≷ VK, so that condition 30b is satisfied.

If a-r+s-t = 0 and k > i = j : use

tr (105) (y+z,-y,x+y,w) : r,s,t,k,i-j+k,i.

Thus A-S+R-T = 0 and I > J = K, so that condition 30c is satisfied.

If a-r+s-t = 0 and a+s+i+k = 0 : use

tr (106) (y+z+w,-y,x+y+w,w) : r,s,t,-i,-j,-k.

V = 1 so that condition 30b is satisfied.

If a+r-s-t = 0 and vk ≥ vi > vj : use

tr (107) (y-z,-y,-x-y,w) : r,s,t,-k,i-j-k,-i.

Thus $A+R-S-T = 0$, $VI \neq VK$ and $VI > VJ$, so that conditions 29-IV and 30c are satisfied.

If $a+r-s-t = 0$ and $k > i = j$: use

tr (108) $(-x, x+z, x+y, w)$: $r, s, t, -i+j+k, k, j$.

Thus $I = J > K$, so that condition 30c is satisfied.

If $a+r-s-t = a+r+j+k = 0$: use

tr (109) $(-x, x+z+w, x+y+w, w)$: $r, s, t, -i, -j, -k$.

Thus $V = 1$, so that condition 30-c1 is satisfied.

Condition 31. If $a = b$ and $|r|=|s|$ and (a) $vj \geq vi$; if $j = i$ and (1) $r = s$, then $v = -1$; (2) $r = -s$, then $s \leq 0$: (b) if $k = 0$, then $r \geq 0$: use

tr (110) $(-y, -x, -z, w)$: $s, r, t, -j, -i, -k$.

Thus $VI \geq VJ$ and if $I = J$, then either $S+R = 0$, $S \geq 0$ or $R = S$ and $V = 1$ and condition 31 is satisfied.

If $a = b$, $r+s = 0$, $vj > vi$ and $k = 0$: use

tr (111) $(y, x, -z, w)$: $r, s, t, j, i, -k$.

Thus $VI > VJ$ and $S > 0$ in accordance with conditions 8 and 31.

If $a = b$, $r = s$, $a+t-vi-vj = 0$ and $2vk < s+r$: use

tr (112) $(-y-vw, -x-vw, -z, w)$: $s, r, t, i, j, vs+vr-k$.

Thus $A+T-VI-VJ = 0$ and $2VK > S+R > 0$, so that condition 31c is satisfied.

If $a = b$, $r = -s$, and $a+t+i-j = 0$: use

tr (113) $(y+w, x+w, -z, w)$: $r, s, t, -i, -j, -k$.

Condition 31c is satisfied.

If $a = b$, $r = s$, $c-2vk = 0$ and $vi+vj < s$: use

tr (114) $(-y, -x, -z-vw, w)$: $r, s, t, vr-j, vs-i, k$.

Thus $C-2VK = 0$ and $VI+VJ > S > 0$ so that condition 31b is satisfied.

If $a = b$, $s = -r > 0$ and $c-2vk = 0$ and if (I) (a) $vi \geq s$; if $vi = s$, then $t > 0$: (b) $vi < -r+vj$: use

tr (115) $(y, x, -z-vw, w)$: $r, s, t, -vr+j, -vs+i, k$.

Thus $C-2VK = 0$, $VI > S+VJ$, so that condition 31b is satisfied.

And if $a = b$, $s = -r > 0$ and $c-2vk = 0$ and if (II) (a) $s \geq vi$; if $s = vi$, then $t \leq 0$: use

tr (116) $(y, -x, z-vw, w)$: $-r, s, -t, vr-j, vs-i, k$.

Since $t \leq 0$ and $r < 0$, it follows from condition 8b that $vj > 0$. Thus $R > 0$, $T \geq 0$ and $VI > S > 0$ so that condition 30b is satisfied.

If $a=b$, $r=-s$, $a+c+2s-2vi-2vk=0$ and (a) $t+r-vi-vj\geq 0$: use

tr (117) $(-y-vw,-x,-z-vw,w)$: $s,r,t,vt+vr-j,va+vs-i,$ $vc+vs-k$.

Thus $B+C+2R-2VJ-2VK=0$. Now $a-r+s-t>0$. Were $a-r+s-t=0$, then by condition 26, $vj>vi$, contrary to condition 31. Thus $2VI-S-T=t+r-2vj\leq 0$, by (a) and $A+R-VI-VJ=-t-r+vj+vi\leq 0$. Now since $a+s-2vi<0$, by condition 13a, and $t+r-vi-vj\geq 0$, therefore $VI\geq vi>VJ>vj$ so that condition 31e is satisfied.

If $a+c+2s-2vi-2vk=0$, $a=b$, $r=s$ and (a) $-t-r+vj\geq vi$: (b) if $-t-r+vj=vi$, then $a+s-vi-vj\geq 0$: (c) $vi>vj$: use

tr (118) $(y-vw,-x,z-vw,w)$: $s,-r,-t,vt-vr-j,-va-vs+i,$ $-vc-vs+k$.

Now $B+C+2R-2VJ-2VK=0$. By (b) and (c), $T+S>0$, $2VI-S-T=-t-r+2vj>0$, so that condition 14 is satisfied. Also $A+R-VI-VJ=t+r-vj+vi\leq 0$, by (b). If $t+r-vj+vi=0$, then $A+R-VI-VJ=0$ and $-S-T+VI=vj$. By (b), $a+s-vi-vj\leq 0$ so that $VJ\geq -S-T+VI\geq 0$. Note that if $|I|=|i|$, $|J|=|j|$, then by (c), $a+s-2vi<0$, therefore $|K|>|k|$. Conditions 27 and 31e are satisfied.

If $b+c+2r-2vj-2vk=0$ and (a) $a+r-vi-vj>0$: (b) $t+s-vi\geq 0$ and if $t+s-vi=0$, then $t\geq 0$: use

tr (119) $(-y,-x-vw,-z-vw,w)$: $s,r,t,va+vr-j,vt+vs-i,$ $vc+vr-k$.

Now, $A+C+2S-2VI-2VK=0$ and $VI+VJ>R+T>0$ and if $J=0$, then $T\geq 0$, and if $T=0$, then $S>0$. Therefore condition 31d is satisfied.

If $b+c+2r-2vj-2vk=0$, $a=b$, $r+s=0$ and (a) $a+r-vi-vj\geq 0$: (b) $t+s-vi\leq 0$; if $t+s-vi=0$, then $t<0$: (c) if $a+r-vi-vj=0$, then $t+s-vi+vj<0$: use

tr (120) $(-y,x-vw,z-vw,w)$: $-s,r,-t,-va-vr+j,vt+vs-i,$ $-vc-vr+k$.

Were $s<0$, then $i=j$, $a+s-2vi\leq 0$, therefore $s>0$ and by (a) and (c), $vi>vj$. We have $A+C+2S-2VI-2VK=0$ and $A+S-2VI=-(a+r-2vj)<0$. Now $-R-T+VJ=vi\leq VI$, by (a). Therefore conditions 13 and 31d are satisfied.

If $a+b+c+2r+2s+2t-2vi-2vj-2vk=0$, $a=b$ and $r=s$ and (a) $a+t+r-vi-vj>0$: use

tr (121) $(-y-vw,-x-vw,-z-vw,w)$: $s,r,t,va+vt+vr-j,va+vt+vs-i,vc+vr+vs-k$.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK=0$ and $A+T+R-VI-VJ<0$.

Condition 31f is therefore satisfied.

If $a = b$, $|r|=|s|$ and $a+b+c-2r+2s-2t-2vi+2vj-2vk = 0$ and (a) $a-r-t-vi+vj > 0$: (b) $a-r-t-vi \leq 0$: use

tr (122) $(y-vw, x+vw, -z-vw, w)$: $-s, -r, t, va-vt-vr+j, -va+vt-vs+i, vc-vr+vs-k$.

Thus $A+B+C-2R+2S-2VI+2VJ-2VK = 0$. By (a), $A-T-R-VI+VJ < 0$ and condition 31g is satisfied.

If $a = b$, $|r|=|s|$ and if $a+b+c-2r+2s-2t-2vi+2vj-2vk = 0$ and (a) $a-r-t-vi > 0$: use

tr (123) $(y-vw, -x+vw, z-vw, w)$: $s, -r, -t, -va+vt+vr-j, -va+vt-vs+i, -vc+vr-vs+k$.

Thus $A+B+C+2R+2S-2T-2VI-2VJ-2VK = 0$ and $A+T+R-VI-VJ = -a+t+s-vj+vi < 0$, so that condition 31e is satisfied.

If $a = b = -2r = -2s = 2t$ and $a+c+2s-2vi-2vk = 0$ and (a) $4vi < a+2vj$: use

tr (124) $(-x-vw, x+y-z, -z-vw, w)$: $r, s, t, vt-i+j, j, vc-k+vs-j$.

Thus $A+C+2S-2VI-2VK = 0$. By (a), $4VI > A+2VJ$. Were $K = 0$, then $c+s-vk-vj = 2vi+2vk-vk+s-vj = 0$. And $4vi+2vk = a+2j = c+2vi$. This implies that $a = 2i = 2j = c$ and therefore $k = 0$, $s < 0$, which is contrary to 8c. Therefore conditions 8 and 31b are satisfied.

Condition 32. If $a = b = c$, $|r|=|s|=|t|$ and $a+s-vi-vk = 0$ and (a) $r+t > 2vj$: use

tr (125) $(-z-vw, -y, -x-vw, w)$: $t, s, r, i, vt+vr-j, k$.

Thus $A+S-VI-VK = 0$ and $2VJ > R+T > 0$, so that condition 32a is satisfied.

If $a+s+i+k = 0$ and $r+t = 0$: use

tr (126) $(z+w, -y, x+w, w)$: $r, s, t, -i, -j, -k$.

So $V = 1$ and condition 32a is satisfied.

If $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$ and (a) $a+t+s-vi-vk > 0$: use

tr (127) $(-z-vw, -y-vw, -x-vw, w)$: $t, s, r, va+vr+vs-k, va+vr+vt-j, va+vs+vt-i$.

By (a) and condition 31f, $a+t+s-vi-vk \geq 0$ and $a+t-r-vi-vj < 0$, so that $vi \neq vj > vk$. By (a), $A+T+S-VI-VK < 0$. Since $VJ \neq VK$, it follows that $A+T+S-VI-VJ < 0$, thus condition 32b is satisfied.

If $a = b = c = s = r = -t$, $3a-2r-2s+2t-2vi-2vj+2vk = 0$ and if (I) (a) $a-r-s-vi+vk > 0$: (b) $a-r-s-vi \leq 0$: use

tr (128) $(-z-vw, -y-vw, x+vw, w)$: $r, s, t, va-vr-vs+k, va+vt-vr-j, -va-vt+vs+i$.

Thus $A+B+C-2R-2S+2T-2VI-2VJ+2VK = 0$ and $A-R-S-VI+VK < 0$, so

that condition 32c is satisfied.

And if $a=b=c=s=r=-t$, $3a-2r-2s+2t-2vi-2vj+2vk=0$ and if (II) $a-r-s-vi>0$: use

tr (129) $(z-vw, y-vw, -x+vw, w)$: $-r, -s, t, -va+vr+vs-k, -va+vt+vr-j, -va-vt+vs+i$.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK=0$ and $A+R+S-VI-VK=-(a-2s+vk-vi)<0$, therefore condition 32b is satisfied.

Condition 33. If $c=d$ and (a) $|i|\geq|s|$: (b) if $|i|=|s|$, then $|j|>|r|$: use

tr (130) $(-x, -y, w, -z)$: $j, i, t, -s, -r, -k$.

Thus $|S|\geq|I|$, and if $|S|=|I|$, then $|R|>|J|$, so that condition 33a is satisfied. If necessary, by use of the proper one of transformations (1), (2) and (3), the form may be brought into conformity with condition 8 without altering the absolute values of the coefficients.

If $c=d$ and $|s|=|i|$, $|r|=|j|$ and if (I) (a) $vr>0$, $vs<0$ and $t>0$: use

tr (131) $(-x, y, w, z)$: $j, -i, -t, -s, r, k$.

Thus $SR<0$, $T<0$. If $k=0$, then $v=-1$ and $S>0$ since $s>0$. So that conditions 8 and 33b are satisfied.

And if $c=d$ and $|s|=|i|$, $|r|=|j|$ and if (II) (a) $vr<0$, $vs>0$ and $t>0$: use

tr (132) $(x, -y, w, z)$: $-j, i, -t, s, -r, k$.

As before, $SR<0$, $T<0$. If $k=0$, then $s=i=S>0$ and conditions 8 and 33b are satisfied.

And if $c=d$ and if $|s|=|i|$, $|r|=|j|$ and if (III) $r\leq 0$, $s\leq 0$, $v=-1$, and if $s=0$, then $r<0$: use

tr (133) $(-x, -y, w, -z)$: $j, i, t, -s, -r, -k$.

Now $V=1$, since $s\leq 0$, $r\leq 0$, $k<0$ and conditions 8 and 33c are satisfied.

And if $c=d$ and $|s|=|i|$, $|r|=|j|$ and if (IV) $s\geq 0$, $r\geq 0$, $v=-1$ and if $s=0$, $r>0$: use

tr (134) $(x, y, w, -z)$: $-j, -i, t, s, r, -k$.

Now $V=1$, since $s=S\geq 0$. If $s=0$, then $R>0$, conditions 8 and 33c are satisfied.

If $c=d$ and $a-2t=0$ and (I) (a) $s=i$: (b) $s>j+r$: use

tr (136) $(x+y, -y, w, z)$: $i-j, i, t, s, s-r, k$.

Thus $A-2T=0$, $S=I$ and $|S|>|J+R|$, so that condition 33d is satisfied.

If $c=d$ and $a-2t=0$ and (II) (a) $-s=i$: (b) $|s|=|j-r|$: use

tr (137) $(-x-y,y,w,z) : -i+j,-i,t,-s,-s+j,k.$

Thus $A-2T=0$, $S=-I$ and $|S|>|J\pm R|$, so that condition 33d is satisfied.

If $a-2t=0$, $a+c+2s-2vi-2vk=0$, $a+3s=0$ and if (I) (a) $|s|=|i|$: (b) $2vj\geq|s|$: (c) $vj\leq vi$: use

tr (138) $(-x-y+z,y,z,-vz+w) : r-vj+a/6,s,t,-i,-i+j,-k.$

Thus $A+C+2S-2VI-2VK=0$ and $A-2T=0$, $A+3S=0$, $2VJ\geq|S|$, so that condition 33-dl is satisfied.

And if $a-2t=0$, $a+c+2s-2vi-2vk=0$, $a+3s=0$ and if (II) (a) $|s|=|i|$: (b) $vj>vi$: use

tr (139) $(vx-vy-vz,vy,-vz+w,z) : -r+vj-a/6,s,-t,-i,vr-vs,-k.$

Thus $A+2T=0$ and $A+C+2S-2VI-2VK=0$, and in accordance with condition 33e, $|R|\geq VI+VJ$ since $R-VI-VJ=vj-a/6>0$, since $vj>vi=a/3$.

If $a+2t=0$ and (I) (a) $s=\rho i$, $\rho=\pm 1$: (b) $|r|\leq|s|$: use

tr (140) $(-x-y,-y,w,-z) : \rho(j+i),\rho i,-t,-\rho s,-\rho r-\rho s,-k.$

Thus $A-2T=0$ and $|S|=|I|$ and $|R|+|J|>|S|$. So that condition 33d is satisfied. Note that if $k=0$, then $s>0$ and therefore $S>0$.

And if $a+2t=0$ and (II) (a) $s=\rho i$, $\rho=\pm 1$: (b) $|i+j|>|r|>|s|$: use

tr (141) $(x-y,-y,w,z) : \rho i,-\rho j-\rho i,t,\rho s,-\rho s-\rho r,k.$

Thus $A+2T=0$ and $|R|>|I+J|$, so that condition 33e is satisfied.

Condition 34. If $a=b$, $c=d$ and (a) $|j|\geq|s|$: (b) if $|j|=|s|$, then $|r|>|i|$: use

tr (142) $(y,x,w,z) : i,j,t,r,s,k.$

Thus $|S|\geq|J|$ and if $|J|=|S|$, then $|R|>|I|$. If condition 8 is not satisfied, then the use of the proper one of the transformations (1),(2) and (3) will satisfy condition 8 and since absolute values are unchanged by the latter transformations, condition 34 is established.

If $a=b$, $c=d$, $|s|=|j|>0$, $|r|=|i|>0$ and if (I) (a) $vk>0$ and $j+s=0$, $r=i$ and $t\geq 0$ or (b) $k=0$, $j=s$, $r+i=0$ and $t>0$: use

tr (143) $(y,-x,w,-z) : -i,j,-t,-r,s,-k.$

And if $a=b$, $c=d$, $|s|=|j|>0$, $|r|=|i|>0$ and if (II) (a) $vk>0$, $j=s$, $r+i=0$ and $t\geq 0$ or (b) if $k=0$, then $j=-s$, $r=i<0$ and $t>0$: use

tr (144) $(-y,x,w,-z) : i,-j,-t,r,-s,-k.$

In either case, $|J|=|S|$, $|R|=|I|$, $SR<0$ and either $T<0$ or $T=0$, $K\geq 0$ so that condition 34 is satisfied.

If $a=b$, $c=d$, $b+c+2r-2vj-2vk=0$ and (I) (a) $a+r-vj\geq|s|$ and $vj>|r|$: (b) if $a+r-vj=|s|$, then $|t+s-vi|\geq|r|$. If the latter equality holds, then $vj>vi$: use

tr (145) $(-y,-x+z,z,-vz+w)$: $-t-s+vi,-a-r+vj,t,-j,-i,-vc+k+j$.

Thus $A+C+2S-2VI-2VK=0$ and by (a), $A+S-2VI<0$. By (a) and (b), $|S|\geq|T+R-VJ|$ and if $|S|=|T+R-VJ|$, then $|R|\geq A+S-VI$. If both equalities hold, then by (b), $VI>VJ$ so that condition 34c is satisfied.

And if $a=b$, $c=d$, $b+c+2r-2vj-2vk=0$ and (II) (a) $a+r-vj\geq s>0$ and $|r|\geq vj$: (b) if $a+r-vj=|s|$, then $|t+s-vi|>|r|$: use

tr (146) $(vy,-vx-vz,vz+w,-vz)$: $t+s-vi,-a-r+vj,-t,-vr,vs,vc-k+vr$.

Thus $A+C+2S-2VI-2VJ=0$. By (a) and (b), $|S|\geq|J|$ and if $|S|=|J|$, then $|R|>|I|$, so that condition 34 is satisfied. By (a), $|r|\geq vj$. Accordingly, $A+S-2VI\leq 0$. If $A+S-2VI=0$, then $|r|=|j|$ and $T+R=s-vi\geq 0$. If $|r|=|j|$ and $|s|=|i|$, then $t<0$, by condition 34a, since $rs<0$. Hence, $T>0$. Now $2VJ-T-R=s+vi>0$ and condition 13 is satisfied.

And if $a=b$, $c=d$, $b+c+2r-2vj-2vk=0$ and (III) (a) $a+r-vj\geq -s>0$ and $|r|\geq vj$: (b) if $a+r-vj=-s$, then $|t+s-vi|>|r|$: use

tr (147) $(vy,vx-vz,-vz+w,z)$: $-t-s+vi,-a-r+vj,t,vr,vs,-vc+k-vr$.

Thus $A+C+2S-2VI-2VK=0$ and by (a) and (b), $|S|\geq|J|$ and if $|S|=|J|$, then $|R|=|I|$, so that condition 34a is satisfied. By (a), $A+S-2VI\leq 0$ and $2VJ-T-R=-s-vi>0$. Were $-s-vi=0$, then since $a+c+2s-2vi-2vk\geq 0$ and $a+c+2r-2vj-2vk=0$, would $vj-r\geq -s+vi$. That is $|j|=|r|=|s|=|i|$. But $vi\geq|t+s-vi|>|r|$, by (b), a contradiction.

And if $a=b$, $c=d$, $b+c+2r-2vj-2vk=0$ and (IV) (a) $|s|=|j|$, $|r|=|i|$: (b) $a+r-vi-vj>0$: (c) $t+s-vi<0$: use

tr (148) $(y,x-vw,-vw,-vz+w)$: $-vi,-vj,t,-va-vr+j,-vt-vs+i,-vc+k+j$.

Were $s<0$, then since $a+c+2r-2vj-2vk=0$ and $|j|=|s|$, $|i|=|r|$ then $a+c+2s-2vi-2vk=0$. Hence $a+s-2vi=a+r-vi-vj\leq 0$, contrary to (b). Therefore, $s>0$ and since $rs<0$, must $t\leq 0$, by condition 34a. Thus $A+C+2S-2VI-2VK=0$ and $|S|=|T+R-VJ|$ and $|R|=A+S-VI$, by (a). By (b), $|I|>|J|$, so that condition 34c is satisfied.

And if $a=b$, $c=d$, $b+c+2r-2vj-2vk=0$ and (V) (a) $|s|=|j|$, $|r|=|i|$: (b) $a+r-vi-vj>0$: (c) $t+s-vi\leq 0$: use

tr (149) (y,-x-vw,-vw,vz+w) : vi,-vj,-t,va+vr-j,-vt-vs+i, vc-k-j.

Thus $A+C+2S-2VI-2VK=0$, $|S|=|T+R-VJ|$, $|R|=A+S-VI$ and $|I|>|J|$, so condition 34d is satisfied. If $t+s-vi=0$, then $t<0$, $s>0$. Were $s<0$, then by (a), $a+c+2s-2vi-2vk=0$, hence $a+s-2vi=a+r-vi-vj\leq 0$, contrary to (b). Since $t<0$, it follows that $T>0$. If $J=0$, then 8b is satisfied.

If $a=b$, $c=d$ and $a+c+2s-2vi-2vk=0$ and (I) (a) $|t+r-vj|\geq|s|$ and $|j|\geq|r|$ and if $|j|=|r|$, then $|s|=|i|$ and $t>0$: (b) if $|t+r-vj|=|s|$, then $a+s-vi\geq|r|$. If both equalities hold, then $vj>vi$. Use

tr (150) (y-z,x,-z,vz+w) : -a-s+vi,-t-r+vj,t,j,i,vc-k-i.

Thus $A+C+2R-2VJ-2VK=0$ and by (a), $2VI\geq T+S\geq 0$ and if $T+S=0$, then $A+R-2VJ=0$ and $T>0$. If $2VI=T+S$, then $A+R-2VJ=0$. By (a) and (b), $|S|\geq A+R-VJ$ and if $|S|=A+R-VJ$, then $|R|\geq|T+S-VI|$ and if both equalities hold, then $|I|>|J|$, so that condition 34b is satisfied.

And if $a=b$, $c=d$ and $a+c+2s-2vi-2vk=0$ and (II) if $a+c+2s-2vi-2vk=0$ and (a) $-r\geq vj$: (b) $|t+r-vj|\geq|s|>vi$ and if equality holds, then $a+s-vi>|r|$: use

tr (151) (vy-vz,vx,-vz+w,z) : -a-s+vi,-t-r+vj,t,vr,vs, -vc+k-vs.

Thus $A+C+2R-2VJ-2VK=0$. By (b), $|S|\geq|J|$ and if $|S|=|J|$, then $|R|>|I|$. By (a), $2VI\geq|S+T|>0$ and since $|s|\geq|i|$, $A+R-2VJ<0$, so that conditions 34a and 34b are satisfied.

And if $a=b$, $c=d$ and $a+c+2s-2vi-2vk=0$ and (III) if $a+c+2s-2vi-2vk=0$ and (a) $r\geq vj$: (b) $|t+r-vj|\geq|s|$ and if $|t+r-vj|=|s|$, then $a+s-vi>|r|$: use

tr (152) (-vy+vz,vx,vz+w,-z) : -a-s+vi,t+r-vj,-t,vr,-vs, vc+vs-k.

Thus $A+C+2R-2VJ-2VK=0$ and $|S|\geq|J|$. And if $|S|=|J|$, then $|R|>VI$. Therefore, conditions 34a and 34b are satisfied.

If $a=b$, $c=d$ and $a+b+c+2r+2s+2t-2vi-2vj-2vk=0$ and (I) (a) $a+t+r+s-vj\geq 0$: (b) if $a+t+r+s-vj=0$, then $vj>vi$: (c) $vj>|r|$: use

tr (153) (y-z,x-z,-z,vz+w) : -a-t-s+vi,-a-t-r+vj,t,j,i, vc-k-j-i.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK=0$ and by (c), $A+T+S-2VI<0$ and by (a) and (b), $|S|\geq|A+T+R-VJ|$ and if $|S|=|A+T+R-VJ|$, then $|R|>|A+S+T-VI|$, so condition 34d is satisfied.

And if $a=b$, $c=d$ and $a+b+c+2r+2s+2t-2vi-2vj-2vk=0$ and (II) (a) $|a+t+r-vj|\geq|s|$ and if equality holds, then $|a+t+s-vi|>|r|$: (b) $-r\geq vj$: use

tr (154) $(vy-vz, x-vz, -vz+w, z)$: $-a-t-s+vi, -a-t-r+vj, t, vr, vs, -vc-vr-vs+k$.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK=0$. By (b), $A+T+S-2VI\leq 0$. Since $|s|>|j|$, $A+T+R-2VJ<0$ so that conditions 8 and 34d are satisfied. Were $r\geq vj$, then $a+2t+2r-2vj=0$ and $a+2t=0$, $a+c+2s-2vi-2vk=0$ and since $|s|\geq|t+r-vj|=-t$, we should have $a+t+s=0$, contrary to condition 20a.

If $a=b$, $c=d$, $a+b+c-2r+2s-2t-2vi+2vj-2vk=0$: use

tr (155) $(-vy+vz, vx-vz, vz+w, -z)$: $-a+t-s+vi, -a+t+r-vj, -t, vr, -vs, vc-vr+vs-k$.

Thus $A+B+C+2R+2S+2T-2VI-2VJ-2VK=0$. By conditions 29-IIIb and 33, $a-r+s-t>0$. Obviously, $A+T+S-2VI<0$ and $|S|>|J|$.

If $a=b$, $c=d$, $c-2vk=0$ and (I) (a) $s\leq -r+vj$, if $s=-r+vj$, then $vj>vi$: (b) $vj+r>0$: use

tr (156) $(y, x, -z, vz+w)$: $-s+vi, -r+vj, t, j, i, k$.

Thus $C-2VK=0$. By (a), $S\geq -R+VJ$ and if $S=-R+VJ$, then $VI>VJ$. By (b), $2VI$ S. Therefore conditions 12 and 34e are satisfied.

And if $a=b$, $c=d$, $c-2vk=0$ and (II) (a) $s<-r+vj$: (b) $vj+r\leq 0$: use

tr (157) $(vy, -vx, vz+w, -z)$: $s-vi, -r+vj, -t, -vr, vs, k$.

Thus $C-2VK=0$, $S>0$, $R\geq 0$. By (b), $2VI\geq S$. By (a), $S>VJ$. If $2VI=S$, then $2VJ>R\geq 0$. If $2VI=S$ and $R=0$, then $-r=vj$, $s=vi$ and by condition 33a, $t\leq 0$ and therefore $T\geq 0$. It follows that conditions 12 and 34 are satisfied.

If $a=b$, $c=d$ and (I) $a-2t=0$ (a) $|s|=|j|$: (b) $|s|\geq vi+r$; if $s=vi+r$, then $vi>r$: use

tr (158) $(vy, vx-vy, w, z)$: $vi-vj, vj, -t, vr, vs-vr, k$.

Thus $A+2T=0$ and by (a), $|S|=VI+VJ$. By (b), $|R|\geq|I|$, so that condition 34i is satisfied. By (b), $2|R|>|S|$ so that condition 21-I is satisfied.

And if $a=b$, $c=d$ and (II) (a) $-s=vj$: (b) $vi-r\leq -s$; if $vi-r=-s$, then $vi<-r$: use

tr (159) $(-vy, -vx+vy, w, z)$: $vj-vi, -vj, -t, -vr, -vs+vr, k$.

Thus $A+2T=0$ and by (a), $|S|=VI+VJ$. By (b), $|R|\geq|I|>|J|$, so that condition 34i is satisfied. By (b), $2|R|>|S|$ so that condition 21-I is satisfied.

If $a=b$, $c=d$ and $a+2t=0$ and (I) (a) $|s|\leq vi+vj$; if $|s| = vi+vj$, then $|r|< vi$: (b) $vi\geq vj$: use

tr (160) $(vx+vy,vx,w,z)$: $vi,vi+vj,-t,vs+vr,vs,k$.

Thus $A-2T=0$, $|S|=|J|$. By (a), $2|R|\geq|S|$ and if $|J|=|S|$, then $VI+|R|>|S|$. So conditions 34f and 21-I are satisfied.

And if $a=b$, $c=d$ and $a+2t=0$ and (II) (a) $0<-s\leq vi+vj$ and $vi\geq vj$; if $-s=vi+vj$, then $|r|<|i|$: use

tr (161) $(-vx-vy,-vx,w,z)$: $-vi,-vi-vj,-t,-vr-vs,-vs,k$.

Thus $A-2T=0$, $|S|=|J|$ and by (a), $2|R|\geq|S|$. By (b), $VI+|R| >|S|$. Now, $|R|>|I|$ since $2|R|>|S|$. Therefore condition 34f is satisfied.

And if $a=b$, $c=d$ and $a+2t=0$ and (III) (a) $0<s\leq vi+vj$; if $s=vi+vj$, then $|r|<vj$: (b) $vj>vi$: use

tr (162) $(vx,vx-vy,w,z)$: $-vj,vj+vi,t,vs+vr,-vr,k$.

Thus $A+2T=0$ and by (a), $|S|\geq VI+J$. By (a), $2|R|>|S|$ and by (b), $|R|>|J|>|I|$, so that condition 34f is satisfied.

And if $a=b$, $c=d$ and $a+2t=0$ and (IV) (a) $vi+vj\geq -s>0$; if $vi+vj=-s$, then $r<vj$: (b) $vj>vi$: use

tr (163) $(-vx,-vx+vy,w,z)$: $vj,-vi-vj,t,-vs-vr,vr,k$.

If $a=b$, $c=d$, $a-2t=b+c+2r-2vj-2vk=0$ and (I) (a) $t+vi \geq -r+vj$ and if $t+vi=-r+vj$, then $vi-r<t$: use

tr (164) $(-x,-x+y-z,-z,vz+w)$: $-a-r+vj,-t-r-vi+s+vj,t,j-i,j,$ $vc-k-j$.

Thus $B+C+2R-2VJ-2VK=0$ and $A-2T=0$ by (a). And $-R+VJ\geq T +VI$ and if $T+VI=-R+VJ$, then $-R+VI>T$. So condition 34g is satisfied. Note that since $r<0$, $a-2t=0$, then $s<0$ and since $|s|\geq|r|$, $vj\geq vi$ for $a+c+2s-2vi-2vk\geq 0$ and $a+c+2r-2vj-2vk=0$.

And if $a=b$, $c=d$, $a-2t=b+c+2r-2vj-2vk=0$ and (II) (a) $a+r+s-vj=0$: (b) $-r+t\leq -2s+vi$: use

tr (165) $(y,-x-y+z,z,-vz+w)$: $2s+t-vi,s,t,-j,-j+i,-vc+k+j$.

Thus $A+C+2S-2VI-2VK=0$, $A-2T=0$. Now $-2S+VI=T-R+VJ$ and $A+S+R-VI=-r+t+2s-vi\leq 0$, by (b). So condition 34h is satisfied.

If $a=b$, $c=d$, $a-2t=0$, $a+c+2s-2vi-2vk=0$ and (I) (a) $-2s +vi\leq t-r+vj$; if $-2s+vi=t-r+vj$, then $a+s+r-vi>0$: (b) $vi-r>vj -s$: use

tr (166) $(-x-y+z,x,z,-vz+w)$:$-a-s+vi,-t+r-s+vi-vj,t,-i+j,$ $-i,-vc+k+i$.

Thus $B+C+2R-2VJ-2VK=0$ and $A-2T=0$. Now $-R+VJ-(T+VI)=t +j+s-vi>0$ since $|r|<|s|$, by (b). Were $|r|=|s|$, then $i=j$ and $vi -r=vj-s$, and by (a), $-2s+vi\leq t-r+vj$. Hence afortiori, $t+j+s-vi$

> 0. By (a), $|S| \geq A+R-VJ$ and if $|S| = A+R-VJ$, then $-R+T > -2S+VI$. By (b), $2VI > S+T > 0$. So conditions 14 and 34g are satisfied.

And if $a = b$, $c = d$, $a-2t = 0$, $a+c+2s-2vi-2vk = 0$ and (II)(a) $-2s+vi \leq t-r+vj$; if $-2s+vi = t-r+vj$, then $a+s+r-vi > 0$: (b) $vj-s \geq vi-r$: use

tr (167) $(-vx-vy+vz, vx, vz+w, -z)$: $-a-s+vi, -t+r-s+vi-vj, t, -vs+vr, -vs, vc-k+vs$.

Thus $A-2T = B+C+2R-2VJ-2VK = 0$. As in the preceding case, condition 34g is satisfied. By (b), $2VI \geq S+T > 0$ and if $2VI = S+T$, then $A+R-2VJ < 0$, since $|s| > vi$.

If $a = b$, $c = d$, $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$ and $a+2t = 0$ and (I) (a) $a+r+2s-vj \geq 0$: use

tr (168) $(-x+z, -x-y+z, z, -vz+w)$: $-a-t-r-vj, -a-r-s+vi+vj, -t, -i-j, -j, -vc+i+j+k$.

Thus $A-2T = 0$, $A+C+2S-2VI-2VK = 0$ and $-2S+VI-T+R-VJ = a+r+2s-vi-vj \geq 0$, by (a). If $a+r+2s-vi-vj = 0$, then $A+S+R-VI = (t+vj)+(-2r-s)$, of which each parenthesis is negative, so condition 34h is satisfied.

And if $a = b$, $c = d$, $a+b+c+2r+2s+2t-2vi-2vj-2vk = 0$ and $a+2t = 0$ and (II) if $a+r+s+t-vj \geq 0$: use

tr (169) $(y-z, x+y-z, -z, vz+w)$: $-a-r-s+vi+vj, -a-t-r+vj, -t, j, i+j, vc-i-j-k$.

Thus $A-2T = B+C+2R-2VJ-2VK = 0$. Now, $-R+VJ-T-VI = a+r+s+t-vj \geq 0$ and $VI-R-T = a+r > 0$, so condition 34g is satisfied.

Condition 35. If $b = c = d$, $t = s = -i$ and $b+r+j+k = 0$: use

tr (170) $(x, y, y+w, y-z)$: $r, s, t, -i, -j, -k$.

Thus $B+R-J-K = 0$, $S = T = I$ so condition 35 is satisfied.

Condition 36. $a = b = c = d$ and if $|k| \geq |t|$; if $|k| = |t|$, then $vi > |r|$: use

tr (171) (z, w, x, y) : i, s, k, r, j, t.

Thus $|T| \geq |K|$ and if $|T| = |K|$, then $|R| > |I|$. If necessary, the use of the proper one of transformations (1), (2) and (3) will satisfy condition 8 without changing the absolute value of the coefficients. Thus condition 36 is also satisfied.

If $a = b = c = d$ and $|r| = |i| > 0$, $|k| = |t| > 0$, $rt < 0$, $s > 0$ and (I) (a) if $vj > 0$, then $r = i$, $k = -t$: (b) if $j = 0$, then $r = -i$, $k = t$: use

tr (172) $(z, w, -x, -y)$: $-i, -s, k, -r, -j, t$.

If $a = b = c = d$ and $|r| = |i| > 0$, $|k| = |t| > 0$, $rt < 0$, $s > 0$ and (II) (a) if $vj > 0$, then $r = -i$, $k = t$: (b) if $j = 0$, then $r = i$, $k = -t$:

use

tr (173) $(-z,w,x,-y) : i,-s,k,r,-j,-t.$

In each case, $|R|=|I|$, $|T|=|K|$, $RT<0$, $S>0$. By (b), if $J=0$, then $T>0$, thus conditions 8 and 36a are satisfied.

If $a=b=c=d$ and $|t|=|k|$, $|s|=|j|$, $tk<0$, $r>0$ and (I) (a) if $vi>0$, then $t=k$, $s=-j$: (b) if $i=0$, then $t=-k$, $s=j$: use

tr (174) $(w,-z,y,-x) : -r,j,-k,-i,s,-t.$

If $a=b=c=d$ and $|t|=|k|$, $|s|=|j|$, $tk<0$, $r>0$ and (II) (a) if $vi>0$, then $t=-k$, $s=j$: (b) if $i=0$, then $t=k$, $s=-j$: use

tr (175) $(w,z,-y,-x) : -r,-j,k,-i,-s,t.$

In each case, $|T|=|K|$, $|S|=vj$, $TS<0$, $R<0$. If $I=0$, then $T>0$, thus conditions 8 and 36b are satisfied.

If $a=b=c=d$ and $a+t-vi-vj=0$ and (I) $vk\geq s>0$: use

tr (176) $(-vy+w,-vy,-vz,-x+y) : s+r-vk,vk,t,-i,-j,-vs.$

Thus $A+T-VI-VJ=0$ and $S\overline{>}VK$. Now, $|S+R-VK|=|R|\leq|s|\leq S$. Condition 36c is satisfied.

If $a=b=c=d$ and $a+t-vi-vj=0$ and (II) if $s<0$, $r\leq 0$: use

tr (177) $(vx,vx+w,-vz,-x+y) : -vk,-s-r+vk,t,i,j,-vr.$

Since $s<0$, $r\leq 0$, it follows that $|S|>VK$. Now $S+R-VK=vk<|S|$. Thus condition 34c is satisfied.

If $a=b=c=d$ and $a+t-vi-vj=0$ and (III) (a) if $s<0$, then $vk\geq r\geq 0$: (b) if $s>0$, then $r\geq vk$: use

tr (178) $(-vx,-vx+w,-vz,x-y) : vk,s+r-vk,t,-i,-j,-vr.$

By (a) and (b), $|s+r-vk|\geq|s|$ and if equality holds, then $|r|=vk$, $v=-1$. Thus $|S|\geq|S+R-VK|$ and if equality holds, then $R=VK$ and $V=1$. Condition 36c is satisfied.

If $a=b=c=d$ and $a+t-vi-vj=0$ and (IV) (a) if $s<0$, then $vk\geq r$: (b) if $s>0$, then $r\geq vk$: use

tr (178) $(-vx,-vx+w,-vz,x-y) : vk,s+r-vk,t,-i,-j,-vr.$

By (a) and (b), $|s+r-vk|\geq|s|$ and if equality holds, then $r=vk$, $v=-1$. Thus $|S|\geq|S+R-VK|$ and if equality holds, $R=VK$ and $V=1$. Condition 34c is satisfied.

If $a=b=c=d$ and $a+t-vi-vj=0$ and (V) if $|t|=|s|$, $r=vk<vj$: use

tr (179) $(-y+z,-y,x,vy+w) : -vj,s,-s-r+vk,k,-vt,i.$

Since $r=vk$, it follows from condition 36-c1 that $v=1$. Thus $V=v=1$, $B+R-J-K=S+T=0$. Now, $|T|=|J|>|K|$, so conditions 14 and 36 are satisfied.

If $a=b=c=d$ and $a+t-vi-vj=0$ and (VI) if $a+s+i+k=0$: use

tr (180) $(y+z+w,y,z,-x+y+z)$: $r,s,t,-i,-j,-k$.

Thus $A+T-I-J=A+S-I-K=0$, in accordance with condition 36-c3.

Suppose $3a+2r+2s-2t-2vi-2vj-2vk=0$ and $|k|=|t|$. By condition 34d, $a+t+r+s-vj\leq 0$, therefore $a-2vi-2vk=a+2t-2vi\geq 0$. By condition 15, $a+t+s-2vi\leq 0$, therefore $s=t$, $a+2t-2vi=0$ and $a+t+r+s-vj=0$. By condition 28, $vj=vk=-t$. By conditions 34-d1 and 36, $|r|=|i|=|j|$. It follows that if $|k|=|t|$, then $-r=-s=-t=vi=vj=vk$ and $a+4t=0$, in accordance with condition 36-d1.

If $a=b=c=d$ and $3a+2r+2s+2t-2vi-2vj-2vk=0$ and (I) if $a+t+r+s-vk\geq 0$; if equality holds, then $vk>vi$: use

tr (181) $(-y+z,-y,x-y,vy+w)$: $-a-t-s+vi,s,-a-r-s+vk,k,va-i-j-k,i$.

Now $A+T+R+S-VK=-(a+t+r+s-vk)\leq 0$. If equality holds, then $VI>VK$. Thus conditions 36d and 36-d3 are satisfied. Note that $|T|\geq|t|$ and if $|T|=|t|$, then $|R|>|r|$.

If $a=b=c=d$ and $3a+2r+2s+2t-2vi-2vj-2vk=0$ and (II) if $a+t-vi-vj-vk\geq 0$ and if equality holds, then $-r>vk$: use

tr (182) $(vx,vx+w,vx-vz,-x+y)$: $-vk,-a-r-s+vk,-a+vi+vj+vk,va+vr+vt-j,j,-vr$.

Now, $A+T-VI-VJ-VK=-(a+t-vi-vj-vk)\leq 0$. If equality holds, then $VK>-R>0$. Note that $|T|\geq|t|$, and if $|T|=|t|$, $|S|>|s|$. Thus condition 36-d2 is satisfied.

If $a=b=c=d$ and $3a+2r+2s+2t-2vi-2vj-2vk=0$ and (III) if $a+t-vi-vj-vk\geq 0$ and if equality holds, then $vk>-s>0$: use

tr (183) $(vy+w,vy,vy-vz,x-y)$: $-a-r-s+vk,-vk,-a+vi+vj+vk,i,va+vt+vs-i,-vs$.

Now, $A+T-VI-VJ-VK=-(a+t-vi-vj-vk)\leq 0$. If equality holds, then $-S>VK$. Note that $|T|\geq|t|$ and if $|T|=|t|$, then $|S|>|s|$. Thus condition 36-d2 is satisfied.

If $a=b=c=d$ and $3a+2r+2s+2t-2vi-2vj-2vk=0$ and (IV) if $3a+2r+2s+2t-2vi-2vj-2vk=a+r+s+t-vk=0$ and (a) $a+s-vi-vj-vk\geq 0$: (b) if $a+s-vi-vj-vk=0$, then $vk>|r|$: use

tr (184) $(-vx,-vx+w,-vx+vy,x-z)$: $-vk,-a+vi+vj+vk,-a-r-s+vk,-va-vt-vr+j,vr,-j$.

Now, $A+R+S+T-VK=-(a+r+s-vi-vk)\leq 0$ since $|t|\geq|i|$. Also, $A+T-VI-VK-VJ=-a-r-s-t+vk=0$ and $|S|\geq|s|\geq|j|=|K|$. Now $j\geq|k|$ and therefore $|K|\geq|R|$ since $a+r+t+s-vj\leq 0$, by 34d and $a+r+s+t-vk=0$,

by hypothesis. Therefore 36-d2 is satisfied. $A+R+S+T-K = -(a+s-vi-vj-vk) \leq 0$, by (a). If $a+s-vi-vj-vk = 0$, then $VI-VJ = a+t+2r-vj = -t-vj \geq 0$, since $|t| \geq |j|$. Therefore 34-d1 is satisfied. Were $a+s-vi-vj-vk = 0$, $|k| = |r|$, then $|s| = |t|$ and $|j| > |r|$ since $a+2t+2r = a-2i-2j = 0$ and $|t| > |i|$. Thus $S = T$, and $|J| < |K|$, contrary to condition 28.

If $a = b = c = d$ and $a+r-j-k = 0$, $t = -s = j = k$ and $i < -r$: use

tr (185) $(w,-x+z,z,-y-z)$: $-r,-k,j,-t,-i,-i$.

Thus $A-R+S-T = 0$, $-S = T = -I > -J = K$ and $A+2S-T < 0$. Condition 36 is satisfied.

If $a = b = c = d$ and if $a+r+j+k = a+s+i+k = 0$: use

tr (186) $(x,y,x+y+w,x+y-z)$: $r,s,t,-i,-j,-k$.

Thus $A+R-J-K = A+S-I-K = 0$, in accordance with condition 36f.

Definition. A form which satisfies conditions 1 through 36 is a reduced form.

CHAPTER V
TRANSFORMATIONS POSSIBLE ON REDUCED FORMS

Definition. An admissible transformation is an integral transformation of determinant 1 which replaces a form having a reduced fundamental hyper-parallelopiped by a form which likewise has a reduced fundamental hyper-parallelopiped.

Lemma 1. There exists no admissible transformation which will replace a reduced form by another reduced form in which at least one of t,s,r,i,j,k is changed in absolute value and the first to be changed is increased.

Lemma 2. There exists no admissible transformation which will replace a reduced form by another reduced form in which the absolute values of the coefficients are unchanged but the sign of at least one of them is changed.

Since lemma 1 implies the like result with the word increased replaced by decreased, evidently lemmas 1 and 2 imply

Theorem 19. No two reduced forms are equivalent.

To prove these two lemmas, every possible admissible transformation which is applicable to a reduced form was considered.

An idea of the method employed to prove theorem 19 may be obtained from the following typical cases. All numerical references are to conditions 1-36.

If $a+b+2t-2vi-2vj=0$, consider the transformation

$$(x-vw, y-vw, z, w) : r, s, t, -va-vt+i, -vb-vt+j, -vr-vs+k.$$

Thus $A+B+2T-2VI-2VJ=0$, and since by (11), $a+t-2vi \leq 0$, then $A+T-2VI \geq 0$. If $a+t-2vi=0$, then $2VK \leq S+R \geq 0$. Thus, $|I| \leq |i|$ and if $|I|=|i|$, then $|J|=|j|$ and $|K| \leq |k|$. If $r+s=2vk$, then $V=-1$, contrary to (11).

If $a+b+c+2r+2s+2t-2vi-2vj-2vk=0$, consider the transformation

$$(x-vw, y-vw, z-vw, w) : r, s, t, -va-vb-vs+i, -vb-vt-vr+j, -vc-vr-vs+k.$$

By (15), $a+t+s-2vi \leq 0$ and if $a+t+s-2vi=0$, then $b+t+r-2vj \leq 0$,

therefore, $|I| \leq |i|$ and if $|I| = |i|$, $|J| \leq |j|$, and if $|I| = |i|$, $|J| = |j|$, then $|K| = |k|$. And $V = -v = -1$ and $A+B+C+2R+2S+2I+2J+2K = 0$, $A+T+S+2I = B+T+R+2J = 0$, contrary to (15a). Note that $a+t+s > 0$, by (20) so that $|I| > 0$.

If $a = b = c = d$ and $a+t-vi-vk = 0$, consider the transformation

$$(y+z, y, w, vx-vy).$$

$|T| = |k| \leq |t|$, by (36). $|S| = |i| \leq |s|$, by (33). $|R| = |j| \leq |r|$, by 33 and 33a. Assume $|r| = |j|$, $|s| = |i|$, $|t| = |k|$. Then the lattice point N may be replaced by one of the points 5, 6, 18 and 25 as listed on page 36. Consider these replacements of N in order.

5. By (14), $s+t = 0$, $|I| = |r| < |i|$. Were $|i| = |r|$, then $|s| = |t| = |r| = |i|$ and therefore $|i| = |j| = |k|$ but $t < 0$, $s > 0$, contrary to (28a).

6. By (36-c3), $v = 1$. Change the sign of x_1 getting $RS < 0$, $T > 0$, contrary to (33b).

18. By (30a), $|I| = |i+j-k| \leq |j|$. If $i = j = k$, then $V = -1$, contrary to (31a).

25. $|I| = |k| \leq |i|$. If $i = j$, then $v = 1$, by (30a) and this is an automorph.

By such a method theorem 19 was proved. These transformations yielded a large number of automorphs which together with the reduced forms of determinant ≤ 50 are tabulated in the unpublished portion of this paper.

VITA

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He was instructor in mathematics at the University of Oklahoma from 1925-28. He studied at the University of Chicago during the summer quarter of 1926, the four quarters of 1928-29 and the first three quarters of 1929-30. There he worked chiefly under Professors Dickson, Graves, Slaught and Barnard. He was a graduate assistant in mathematics at the University of Chicago for the fall and winter quarters of 1929-30.

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